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# Reducing a MIMO Control System to a Form with a Relative Degree[1]

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**Abstract**—Some methods of control theory are applicable to a control system only if the conditions of a relative degree is met. These conditions are not always met, but it is sometimes possible to reduce such systems to a form with a relative degree using output transformation. In this work, we propose an algorithm for computing such a transformation that can be applied to discrete-time linear systems with as many inputs as outputs.

*Keywords*: Discrete systems, relative degree, canonical representations of discrete systems.

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## 1. INTRODUCTION

The existence of a relative degree (RD) vector in a system is a necessary condition for using certain methods of control theory. However [1], not all systems satisfy this condition. Still there are some methods that allow us to achieve an RD-compliant form by transforming the initial system [2, 3]. The type of such a transformation depends on the method: the authors of [2] proposed constructing a feedback control (so the entire feedback system would be changed into an RD-compliant form), while in [3] it was suggested that integrators be applied to the inputs of a system. In these works, nonlinear continuous-time control systems were considered.

The possibility of reducing a system to an RD-compliant form using output vector multiplication with a nondegenerate matrix was studied for the linear discrete-time systems in [4, 5]. More general transformations are considered in the present work (a combination of linear matrix output transform and time-shift operators), thereby allowing us to enlarge the class of systems for which the transformation is possible. This method can be used to solve different control theory problems (e.g., an inversion problem where the initial output is known and the transformed output can be calculated).

## 2. STATEMENT OF THE PROBLEM

Let us consider the dynamic discrete-time linear time-invariant system

$$\begin{cases} x^{t+1} = Ax^t + B\xi^t, \\ y^t = Cx^t, \quad t = 0, 1, 2, \ldots, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^l$, $\xi \in \mathbb{R}^l$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times l}$, $C \in \mathbb{R}^{l \times n}$. We assume that $\mathrm{Rg}\,B = \mathrm{Rg}\,C = l < n$.

**Definition 1** [2]. Vector $r = (r_1, r_2, \ldots, r_l)$ is called the relative degree vector for system (1) if

$$C_iB = 0, \quad C_iAB = 0, \quad \ldots, \quad C_iA^{r_i-2}B = 0, \quad C_iA^{r_i-1}B \neq 0,$$

$$|H(r)| = \begin{vmatrix} C_1A^{r_1-1}B \\ \ldots \\ C_lA^{r_l-2}B \end{vmatrix} \neq 0, \tag{2}$$

[1]This article was translated by the authors.

where the $C_i$ are rows of matrix $C$, $i = 1, 2, \ldots, l$.

**Remark 1.** Nonlinear control systems were considered in [2], so there the definition of a relative degree was formulated differently. Since we here consider linear systems, this definition is simplified.

It is true [1] that the conditions of (2) can be inconsistent, so there exist systems with no relative degree. By using some output transformations, however, we can achieve a form where both conditions of the definition are satisfied.

Let us assume, for example, that $y^t$ is the output of system $\{A, B, C\}$, $\tilde{y}^t$ is the output of system $\{A, B, \tilde{C}\}$, and the following dependence is correct:

$$\tilde{y}^t = T_0 y^t + T_1 y^{t+1} + \ldots + T_p y^{t+p}, \quad t = 0, 1, 2, \ldots, \tag{3}$$

where $T_i \in \mathbb{R}^{l \times l}$, $T_0 \neq 0$, $T_p \neq 0$, $i = 1, \ldots, p$. Then conditions (2) can be satisfied for the second system, but not for the first one. In this case, we say that output transform (3) reduces the system to a form with a relative degree.

Our formal statement of the problem is thus the following: given system $\{A, B, C\}$ with inconsistent conditions in definition (1), we must find the transform of form (3), i.e., find number $p$, matrices $T_i$, $i = 0, 1, \ldots, p$, and matrix $\tilde{C}$ of full rank, such that the outputs of systems $\{A, B, C\}$ and $\{A, B, \tilde{C}\}$ satisfy (3), and the definition of a relative degree is satisfied for the second system.

## 3. LEADING INCOMPLETE RELATIVE DEGREE

The generalization of relative degree that we use below was considered in [4], so let us introduce it now.

**Definition 2.** Vector $r = (r_1, r_2, \ldots, r_l)$ is called the incomplete relative degree (IRD) vector for system (1) if it satisfies the first condition of (2), i.e.,

$$C_i B = 0, \quad C_i A B = 0, \quad \ldots, \quad C_i A^{r_i - 2} B = 0, \quad C_i A^{r_i - 1} B \neq 0.$$

Let us arrange the components of the IRD vector in ascending order (this can always be done by appropriately renumbering the rows of matrix $C$). Suppose that there are $k$ distinct components of this vector. Then

$$r = (r_1^{(1)}, r_2^{(1)}, \ldots, r_{n_1}^{(1)}, \ldots, r_1^{(s)}, r_2^{(s)}, \ldots, r_{n_s}^{(s)}, \ldots, r_1^{(k)}, r_2^{(k)}, \ldots, r_{n_k}^{(k)}),$$

where $r_i^{(p)} = r_j^{(p)}$, $i, j \in \{1, 2, \ldots, n_p\}$; $r_i^{(p)} < r_j^{(p)}$ for $p < q$, $i \in \{1, 2, \ldots, n_p\}$, $j \in \{1, 2, \ldots, n_q\}$ and $n_1 + n_2 + \ldots + n_k = l$. Two types of numbering are therefore introduced for the components of vector $r$: ordinary successive (one-dimensional) numbering and two-dimensional numbering in which each component is specified by the section number (superscript) and its position in the section (subscript). Note that such two-dimensional numbering can also be applied to other $l$ objects, e.g., to the rows of matrix $H(r)$ from (2).

**Definition 3** [4]. Vector $r$ is called the leading IRD (LIRD) of system (1) if $r$ is an IRD vector and rows $\{H_j^{(p)}\}_{j=1}^{n_p}$ are linearly independent for all $p \in \{1, 2, \ldots, k\}$, where $H_j^{(p)}$ is the $j$-th row of section $p$ of matrix $H(r)$.

**Remark 2.** Note that if $|r| = \sum_{i=1}^{l} |r_i| = l$, LIRD vector $r$ is the relative degree vector in the sense of definition 1. Indeed, since components $r_i$ of this vector are positive integers and their sum is equal to $l$, $r_i = 1$, $i = 1, 2, \ldots, l$. It follows that the number of sections $k = 1$ and all rows of matrix $H(r)$ are linearly independent, thereby satisfying the conditions of (2).

The conditions of definition 3 also may not be met, but for a wide range of systems it is possible to satisfy them using the output transform $\tilde{y}^t = Ty^t$, $T \in \mathbb{R}^{l \times l}$, $|T| \neq 0$. This result was also obtained in [4], and we now briefly introduce it for use in further consideration.

**Definition 4** [4]. System (1) is said to be weakly reducible if for each matrix $T \in \mathbb{R}^{l \times l}$ ($|T| \neq 0$) system $\{A, B, T \cdot C\}$ has an IRD vector.

**Lemma 1** [4]. *For any weakly reducible system* $\{A, B, C\}$, *there exists transformation matrix* $T \in \mathbb{R}^{l \times l}$ *such that system* $\{A, B, T \cdot C\}$ *is a form with an* LIRD.

The algorithm for constructing such a transform ($T$) was also given in [4].

## 4. ADDITIONAL STATEMENTS

For convenience in further consideration, we introduce the following notation:

$Dy^t = y^{t+1}$, $y^t \in \mathbb{R}$ is a time-shift operator;

$$V(r)y^t = \begin{pmatrix} D^{r_1-1} & 0 & 0 & \dots & 0 \\ 0 & D^{r_2-1} & 0 & \dots & 0 \\ 0 & 0 & D^{r_3-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & D^{r_l-1} \end{pmatrix} y^t, \quad y^t \in \mathbb{R}^{l \times 1}.$$

Let us investigate how the properties of the system are influenced by applying the time-shift operator.

**Definition 5.** The Rosenbrock matrix for system $\{A, B, C\}$ is a matrix of the structure

$$R(s) = \begin{pmatrix} sI - A & -B \\ C & 0 \end{pmatrix}, \quad s \in \mathbb{C}.$$

Let us denote $\beta(s) = |R(s)|$.

**Lemma 2.** *For the determinant of the Rosenbrock matrix of system* $\{A, B, C\}$, *the following representation outside some circle on the complex plane is valid*:

$$\beta(s) = \frac{\chi_A(s)}{s^l} \begin{vmatrix} \sum_{k=0}^{\infty} \frac{C_1 A^k B}{s^k} \\ \vdots \\ \sum_{k=0}^{\infty} \frac{C_l A^k B}{s^k} \end{vmatrix} \quad \forall s \in \mathbb{C} : |s| > s_0 = \text{const}, \tag{4}$$

*where* $s_0$ *depends only on* $A$. *Here* $\chi_A(s)$ *is the characteristic polynomial of matrix* $A$.

**Proof.** Let $|s| > \lambda_{\max}(A)$, where $\lambda_{\max}(A)$ is the maximum absolute value of the eigenvalues of matrix $A$. Let us add to the $(n+1)$-th row of matrix $R$ the linear combination of the first $n$ rows with coefficients $-C_1(sI - A)^{-1}$, to the $(n+2)$-th with coefficients $-C_2(sI - A)^{-1}$, and to $(n+i)$-th with coefficients $-C_i(sI - A)^{-1}$. We obtain

$$|R(s)| = \begin{vmatrix} sI - A & -B \\ C & 0 \end{vmatrix} = \begin{vmatrix} sI - A & -B \\ 0 & C(sI - A)^{-1}B \end{vmatrix} = \chi_A(s)\left|C(sI - A)^{-1}B\right|.$$

Let us denote $s_0 = \max(\|A\|_1, \lambda_{\max}(A))$, where $\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$ is the matrix norm subordinate to vector norm $\|x\|_1 = \max_j \sum_{i=1}^{n} |x_i|$, $x \in \mathbb{R}^n$. Then $\forall s$: $|s| > s_0$; it follows that $\|As^{-1}\|_1 < 1$ and $(sI - A)^{-1} =$

$s^{-1}(I - As^{-1})^{-1} = s^{-1}\sum_{k=0}^{\infty} A^k s^{-k}$ (the last equality is valid, according to [6]), so $|R(s)| = \chi_A(s)|s^{-1}C \cdot \sum_{k=0}^{\infty} A^k s^{-k} \cdot B|$. Due to the convergence of series $\sum_{k=0}^{\infty} A^k s^{-k}$, we obtain $C \cdot \sum_{k=0}^{\infty} A^k s^{-k} \cdot B = \sum_{k=0}^{\infty} CA^k Bs^{-k}$, so

$$\forall s : |s| > s_0 \quad |R(s)| = \chi_A(s)\left|\frac{1}{s}\sum_{k=0}^{\infty}\frac{CA^kB}{s^k}\right| = \chi_A(s)\frac{1}{s^l}\begin{vmatrix} \sum_{k=0}^{\infty}\frac{C_1A^kB}{s^k} \\ \vdots \\ \sum_{k=0}^{\infty}\frac{C_lA^kB}{s^k} \end{vmatrix}.$$

The proof is complete.

**Corollary.** A system for which $\beta(s) \not\equiv 0$ is a weakly reducible system.

**Proof.** Let us suppose that the assertion is invalid. Then there exists matrix $T \in \mathbb{R}^{l \times l}$ such that system $\{A, B, \tilde{C} = T \cdot C\}$ has no IRD; i.e., $\exists i \in \{1, 2, ..., l\}$: $\tilde{C}_i A^p B = 0, p = 0, 1, 2, ....$ From the lemma, it follows that $\tilde{\beta}(s) = |T|\beta(s) \equiv 0$, which is in conflict with the condition. The corollary is proved.

This corollary allows us to apply Lemma 1 to systems with $\beta(s) \not\equiv 0$. It is these systems that are considered in this work.

Let us examine how matrix $C$ is modified when operator $D$ is applied to the outputs of the system. The following statement describes this:

**Lemma 3.** *If r is the IRD vector for system* $\{A, B, C\}$, *then the outputs of systems* (*with the same initial conditions and inputs*) $\{A, B, C\}$ *and* $\{A, B, \hat{C}\}$, *where*

$$\hat{C} = \begin{pmatrix} C_1A^{r_1 - 1} \\ \vdots \\ C_lA^{r_l - 1} \end{pmatrix},$$

*satisfy equality*

$$\hat{y}^t = V(r)y^t.$$

**Proof.** Let us consider component $\hat{y}_i^t$ of the output vector of the second system: $\hat{y}_i^t = C_iA^{r_i - 1}x^t$. We shall show that $\hat{y}_i^t = \hat{y}_i^{t + r_i - 1}$. Indeed,

$$\hat{y}_i^t = C_iA^{r_i - 1}x^t = C_iA^{r_i - 2}Ax^t = C_iA^{r_i - 2}x^{t+1} - C_iA^{r_i - 2}B\xi^t = C_iA^{r_i - 2}x^{t+1}$$
$$= C_iA^{r_i - 3}x^{t+2} - C_iA^{r_i - 3}B\xi^{t+1} = \ldots = C_ix^{t + r_i - 1} - C_iB\xi^{t + r_i - 2} = C_ix^{t + r_i - 1} = y^{t + r_i - 1}.$$

Here we exclude $\xi_i^{t+p}$ because $r$ is the IRD vector and $C_iA^pB = 0$, $p = 0, 1, 2, ..., r_i - 2$. The lemma is proved.

Applying operator $D$ to the outputs of the system also affects polynomial $\beta(s)$. This follows from the next statement:

**Lemma 4.** *Let there be two systems* $\{A, B, C\}$ *and* $\{A, B, \hat{C}\}$, *where*

$$\hat{C} = \begin{pmatrix} C_1A^{r_1-1} \\ C_2A^{r_2-1} \\ \vdots \\ C_lA^{r_l-1} \end{pmatrix},$$

*and* $r$ *is the IRD vector of the first system. The following equality is then true*: $\hat{\beta}(s) = s^{|r|-l}\beta(s)$, *where* $\hat{\beta}(s)$ *is the determinant of the Rosenbrock matrix of the system* $\{A, B, \hat{C}\}$, $|r| = \sum_{i=0}^{l}|r_i|$.

**Proof.** According to Lemma 2 representation (4) is correct for polynomial $\beta(s)$ $\forall s \in \mathbb{R}$: $s > s_0$. Since $r$ is the system's IRD vector, $C_iA^qB = 0$, $q = 0, 1, \ldots, r_i - 2$. It thus follows that

$$\beta(s) = \frac{\chi_A(s)}{s^l}\begin{vmatrix} \sum\limits_{k=r_1-1}^{\infty}\dfrac{C_1A^kB}{s^k} \\ \vdots \\ \sum\limits_{k=r_1-1}^{\infty}\dfrac{C_lA^kB}{s^k} \end{vmatrix} \quad \forall s > s_0.$$

At the same time, $\forall s > \hat{s}_0$, polynomial $\hat{\beta}(s)$ of system $\{A, B, \hat{C}\}$ is equal to

$$\hat{\beta}(s) = \frac{\chi_A(s)}{s^l}\begin{vmatrix} \sum\limits_{k=0}^{\infty}\dfrac{\hat{C}_1A^kB}{s^k} \\ \vdots \\ \sum\limits_{k=0}^{\infty}\dfrac{\hat{C}_lA^kB}{s^k} \end{vmatrix} = \frac{\chi_A(s)}{s^l}\begin{vmatrix} \sum\limits_{k=0}^{\infty}\dfrac{C_1A^{r_1-1}A^kB}{s^k} \\ \vdots \\ \sum\limits_{k=0}^{\infty}\dfrac{C_lA^{r_l-1}A^kB}{s^k} \end{vmatrix} = \frac{\chi_A(s)}{s^l}\begin{vmatrix} s^{r_l-1}\sum\limits_{k=r_1-1}^{\infty}\dfrac{C_1A^kB}{s^k} \\ \vdots \\ s^{r_l-1}\sum\limits_{k=r_1-1}^{\infty}\dfrac{C_lA^kB}{s^k} \end{vmatrix} = s^{|r|-l}\beta(s).$$

Since $s_0$ depends only on matrix $A$, $s_0 = \hat{s}_0$; i.e. the output values of polynomials $\hat{\beta}(s)$ and $s^{|r|-1}\beta(s)$ equal $\forall s > s_0$, making these polynomials equal.

The lemma is proved.

**Corollary.** If $r$ is the IRD vector of system $\{A, B, C\}$ for which $\beta(s) \not\equiv 0$, then rows $C_1A^{r_1-1}$, $C_2A^{r_2-1}$, ..., $C_lA^{r_l-1}$ are linearly independent.

**Proof.** If we suppose that the given rows are linearly dependent, then

$$\hat{\beta}(s) = \begin{vmatrix} sI - A & -B \\ \hat{C} & 0 \end{vmatrix} \equiv 0,$$

since the last $l$ rows of the Rosenbrock matrix are linearly dependent. According to the lemma, however, $\hat{\beta}(s) = s^{|r|-l}\beta(s)$ $\forall s > s_0$; consequently, $\beta(s) \equiv 0$, which violates the condition. The corollary is proved.

## 5. ALGORITHM FOR CONSTRUCTING THE REQUIRED TRANSFORMATION

The statements formulated above allow us to consider an algorithm for transforming a system with nonzero polynomial $\beta(s)$ to a form with a relative degree.

**Step 1.** Let us consider system $\{A, B, C^{(0)}\}$ for which $\beta^{(0)}(s) \not\equiv 0$. We reduce this system to a form with LIRD $r^{(0)}$, setting $\tilde{C}^{(0)} = T^{(0)} \cdot C^{(0)}$ (which is always possible, according to Lemma 1 and the corollary from Lemma 2).

Let us consider system $\{A, B, \tilde{C}^{(0)}\}$, for which formula $\tilde{\beta}^{(0)}(s) = |T^{(0)}|\beta^{(0)}(s)$ is correct and output $\tilde{y}^{(0)'}$ is associated with the output of the initial system (under the same initial conditions and inputs) by the formula $\tilde{y}^{(0)'} = T^{(0)}y^{(0)'}$. If $r^{(0)}$ is the vector of this system's relative degree, the system has been reduced to the necessary form. If not, we proceed to Step 2.

**Step 2.** Since $r^{(0)}$ is the LIRD vector of system $\{A, B, \tilde{C}^{(0)}\}$, according to Lemma 4 rows $\tilde{C}_1^{(0)}A^{r_1^{(0)}-1}$, $\tilde{C}_2^{(0)}A^{r_2^{(0)}-1}$, ..., $\tilde{C}_l^{(0)}A^{r_l^{(0)}-1}$ are linearly independent, so we may consider system $\{A, B, C^{(1)}\}$, where

$$C^{(1)} = \begin{pmatrix} \tilde{C}_1^{(0)}A^{r_1^{(0)}-1} \\ \tilde{C}_2^{(0)}A^{r_2^{(0)}-1} \\ \vdots \\ \tilde{C}_l^{(0)}A^{r_l^{(0)}-1} \end{pmatrix}.$$

For this system, according to Lemma 4,

$$\beta^{(1)}(s) = s^{|r^{(0)}|-l}\tilde{\beta}^{(0)}(s) = |T^{(0)}|s^{|r^{(0)}|-l}\beta^{(0)}(s),$$

and from Lemma 3 it follows that

$$y^{(1)'} = V(r^{(0)})\tilde{y}^{(0)'} = V(r^{(0)})T^{(0)}y^{(0)'}.$$

Let us reduce it to a form with LIRD $r^{(1)}$ setting $\tilde{C}^{(1)} = T^{(1)} \cdot C^{(1)}$. Thus,

$$\tilde{\beta}^{(1)}(s) = |T^{(1)}||T^{(0)}|s^{|r^{(0)}|-l}\beta^{(0)}(s), \quad y^{(1)'} = T^{(1)}y^{(1)'}.$$

If $r^{(1)}$ is the vector of this system's relative degree, it is reduced to the necessary form. If not, we proceed to the following step:

**Step *p*.** Since $r^{(p-1)}$ is the LIRD vector of system $\{A, B, \tilde{C}^{(p-1)}\}$, we can consider system $\{A, B, C^{(p)}\}$, where

$$C^{(p)} = \begin{pmatrix} \tilde{C}_1^{(p-1)}A^{r_1^{(p-1)}-1} \\ \tilde{C}_2^{(p-1)}A^{r_2^{(p-1)}-1} \\ \vdots \\ \tilde{C}_l^{(p-1)}A^{r_l^{(p-1)}-1} \end{pmatrix}.$$

For this system, $\beta^{(p)}(s) = s^{|r^{(p-1)}|-1}\tilde{\beta}^{(p-1)}(s) = |T^{(p-1)}|s^{|r^{(p-1)}|-1}\beta^{(p-1)}(s) = |T^{(p-1)}||T^{(p-2)}|$ $s^{|r^{(p-1)}|+|r^{(p-2)}|-2l}\beta^{(p-2)}(s) = \ldots = \prod_{q=0}^{p-1}|T^{(q)}| \cdot s^{\left[\sum_{m=0}^{p-1}(|r^{(m)}|-l)\right]}\beta^{(0)}(s)$. Thus, $y^{(p)'} = V(r^{(p-1)})\tilde{y}^{(p-1)'} =$ $V(r^{(p-1)})T^{(p-1)}y^{(p-1)'} = V(r^{(p-1)})T^{(p-1)}V(r^{(p-2)})\tilde{y}^{(p-2)'} = V(r^{(p-1)})T^{(p-1)}V(r^{(p-2)})T^{(p-2)}y^{(p-2)'} = \ldots =$

$\prod_{q=0}^{p-1}(V(r^{(q)})T^{(q)})y^{(0)^t}$. Let us reduce it to a form with LIRD $r^{(p)}$, setting $\tilde{C}^{(p)} = T^{(p)} \cdot C^{(p)}$, where $\tilde{\beta}^{(p)}(s) = |T^{(p)}|\beta^{(p)}(s)$, $\tilde{y}^{(p)^t} = T^{(p)}y^{(p)^t}$. If $r^{(p)}$ is the vector of this system's relative degree, it is reduced to the necessary form. If not, we proceed to step $p + 1$.

**Statement 1.** *If for system* $\{A, B, C^{(0)}\}$ *polynomial* $\beta^{(0)}(s) \not\equiv 0$, *the described algorithm can not work infinitely, and the number of steps will not exceed* $n - l + 1$.

**Proof.** If the algorithm stops at a certain step, the system is reduced to a form with a relative degree (the stop condition is formulated in this manner). Let us suppose that the algorithm can work infinitely. Then at each $k$-th step $|r^{(k)}| > l$, since if $|r^{(k)}| = l$, the conditions of Definition 1 will be satisfied for the system, according to Remark 2. Then $\exists N$: $\sum_{k=0}^{N-1}(|r^{(k)}| - l) > n - l$. Since $\beta^{(N)}(s) = \prod_{p=0}^{N-1}|T^{(p)}| \cdot s^{\left[\sum_{k=0}^{N-1}(|r^{(k)}| - l)\right]}\beta^{(0)}(s)$, $\deg(\beta^{(N)}) > \deg(\beta^{(0)}) + n - l \geq n - l$. However, by considering the representation for $\beta^{(N)}(s)$ via the Rosenbrock matrix of system $\{A, B, C^{(N)}\}$, we find that $\deg(\beta^{(N)}) \leq n - l$. Hence, $\beta^{(N)} \equiv 0$, so $\beta^{(0)}(s) \equiv 0$, which violates the condition. The number of steps is thus finite.

Let us show that number of steps does not exceed $n - l + 1$. At each $k$-th step of the algorithm beginning from the second, the degree of polynomial $\beta^{(k)}(s)$ increases by $|r^{(k-1)}| - l$, compared to the previous step.

The estimate of the number of steps follows because degree of the given polynomial cannot exceed $n - l$.

The statement is proved.

The above algorithm allows us to construct an output transformation of form (3) for the given system while reducing it to a form with a relative degree. The following statement is thus proved:

**Theorem 1.** *For any system* $\{A, B, C\}$ *with polynomial* $\beta(s) \not\equiv 0$ *there exists output transformation* (3) *and matrix* $\tilde{C}$ *of full rank such that outputs of systems* $\{A, B, C\}$ *and* $\{A, B, \tilde{C}\}$ *are associated by formula* (3), *while Definition 1 is correct for the second system.*

## 6. EXAMPLE

Let us consider the system

$$\begin{cases} x^{t+1} = Ax^t + B\xi^t, \\ y^t = Cx^t, \quad t = 0, 1, 2, \ldots, \end{cases}$$

where

$$A = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that this system is controllable and observable. Then

$$C_1B = (1, 0),$$

$$C_2B = (0, 0), \quad C_2AB = (1, 0);$$

i.e., the system has no relative degree, and $r = (1, 2)$ is the LIRD vector of the system. Let us consider system $\{A, B, C^{(1)}\}$, where

$$C^{(1)} = \begin{pmatrix} C_1 \\ C_2A \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

For this system

$$y^{(1)^t} = \begin{pmatrix} y_1^t \\ y_2^{t+1} \end{pmatrix}, \quad \begin{matrix} C_1^{(1)}B = (1, 0), \\ C_2^{(1)}B = (1, 0). \end{matrix}$$

We reduce it to a form with LIRD, setting

$$\tilde{C}_1^{(1)} = C_1^{(1)} = C_1 = (1, 0, 0, 0, 0, 0),$$

$$\tilde{C}_2^1 = C_2^{(1)} - C_1^{(1)} = C_2A - C_1 = (0, 0, 1, 0, 0, 1).$$

Then

$$\tilde{y}^{(1)^t} = \begin{pmatrix} y_1^t \\ y_2^{t+1} - y_1^t \end{pmatrix}, \quad \begin{matrix} \tilde{C}_1^{(1)}B = (1, 0), & \\ \tilde{C}_2^{(1)}B = (0, 0), & \tilde{C}_2^{(1)}AB = (1, 0); \end{matrix}$$

i.e., the system still has no relative degree.

Let us consider system $\{A, B, C^{(2)}\}$, where

$$C^{(2)} = \begin{pmatrix} \tilde{C}_1^{(1)} \\ \tilde{C}_2^{(1)}A \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2A^2 - C_1A \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & -1 & 3 \end{pmatrix}.$$

For this system we have

$$y^{(2)^t} = \begin{pmatrix} y_1^t \\ y_2^{t+2} - y_1^{t+1} \end{pmatrix}, \quad \begin{matrix} C_1^{(2)}B = (1, 0), & \\ C_2^{(2)}B = (1, 0), & C_2^{(2)}AB = (1, 1). \end{matrix}$$

We reduce it to a form with LIRD, setting

$$\tilde{C}_1^{(2)} = C_1^{(2)} = C_1 = (1, 0, 0, 0, 0, 0),$$

$$\tilde{C}_2^2 = C_2^{(2)} - C_1^{(2)} = C_2A - C_1A - C_1 = (0, 0, 2, 1, -1, 3).$$

Then

$$\tilde{y}^{(2)^t} = \begin{pmatrix} y_1^t \\ y_2^{t+2} - y_1^{t+1} - y_1^t \end{pmatrix}, \quad \begin{matrix} \tilde{C}_1^{(2)}B = (1, 0), & \\ \tilde{C}_2^{(2)}B = (0, 0), & \tilde{C}_2^{(2)}AB = (1, 1). \end{matrix}$$

Our $r = (1, 2)$ is thus the vector of this system's relative degree. In other words, the initial system has been reduced to a form with a relative degree.

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## REFERENCES

1. A. V. Il'in, S. K. Korovin, and V. V. Fomichev, *Metody robastnogo obrashcheniya dinamicheskikh system* (Method of Robust Inversion for Dynamical Systems) (Moscow, 2009).
2. A. Isidori, *Nonlinear Control Systems* (Springer-Verlag, London, 1995).
3. C. A. Schwartz, P. W. Gibbens, and M. Fu, "Achieving vector relative degree for nonlinear systems with parametric uncertainties," Int. J. Robust Nonlinear Control **5** (2), 139–151 (1995).
4. A. V. Kraev, A. I. Rogovskii, and V. V. Fomichev, "On a generalization of relative degree," Differ. Equat. **50** (8), 1128–1132 (2014).
5. A. V. Kraev, "Some properties of relative degree of linear time-invariant dynamical systems," in *Nelineinaya Dinamika I Upravlenie* (Fizmatlit, M., 2013), No. 8, pp. 105–112.
6. A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis* (Nauka, M., 1976). 1. A. V. Il'in, S. K. Korovin, and V. V. Fomichev, *Methods of Robust Inversion of Dynamic Systems* (Fizmatlit, Moscow, 2009) [in Russian].