Self-similar solutions for hypersonic source with variable mass loss rate

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Abstract. For the 1D radial hypersonic flow the development of the configuration with two shocks and contact discontinuity is considered. At small and large moments of time solutions in explicit form are found. As follows from these solutions the contact surface accelerates in time. This acceleration makes possible the Rayleigh-Taylor instability to develop. The 2D numerical investigation of the problem has confirmed the instability of the 1D solution.

Keywords: radial hypersonic flow, self-similarity, Rayleigh-Taylor instability

1. Introduction

A model of hypersonic source with variable mass loss rate is widely used in astrophysics presently. The variability of mass loss rate can be interpreted as a Riemann problem for the radial hypersonic flow. We will consider evolution of the flow in the case when configuration with two shocks and contact discontinuity between them takes place. In the related studies (Shidlovskaya, 1976; Chevalier & Imamura, 1983; and others) the flow is considered as self-similar, so that the discontinuities propagate with constant velocities. In our formulation, when there exist some finite radius where the configuration arises from, the problem is not self-similar but permit limiting self-similar solutions at small and large moments of time. The comparison between these solutions reveals the contact surface acceleration, and gives rise a question of the flow stability.

2. Formulation of the problem

Let us consider the exact solution of Euler equations for a stationary radial flow of an ideal gas with zero pressure

$$u = u_S, \quad \rho = A_S r^{-2}, \quad p = 0 \tag{1}$$

We will call this solution for $u_S > 0$ as a hypersonic source, since it describes in the hypersonic approximation the 1D supersonic radial flow (Mises, 1961). Its intensity is defined by two parameters u_S, A_S.

Let the parameters of a source be changed so that for $t = 0$ they are: u_1, A_1 for $r > r_S$ and u_2, A_2 for $r < r_S$. Let $u_2 > u_1$, i.e. stationary flows come into collision. It is natural to expect that for $t > 0$ the flow will consist of two regions of stationary flow

$$u = u_k, \quad \rho = A_k/r^2, \quad p = 0 \tag{2}$$

(hereafter $k = 1, 2$) and of the interaction region between them, which is bounded by two strong shocks and is divided by the contact discontinuity surface on two shock layers. The problem is to describe the flow in the interaction region. There are three independent dimensional parameters : A_1, u_1, r_S in the initial condition, thus, in accordance with the classic theory (Sedov, 1981) the problem is not self-similar. However, for small or large times the last parameter may be inessential, and so the problem can permit self-similar solutions.

It is convenient to choose as independent dimensionless variables

$$x = (r - r_S)/u_1 t, \quad \tau = u_1 t/r_S$$

Let us also introduce dimensionless functions

$$U = u/u_1, \quad R = \rho r^2/A_1, \quad P = pr^2/A_1 u_1^2$$

Then the stationary flows may be represented in the form

$$U = \chi^{k-1}, \quad R = \lambda^{2(k-1)}, \quad P = 0 \tag{3}$$

where $\chi = u_2/u_1$, and $\lambda^2 = A_2/A_1$ are dimensionless parameters of the problem.

In the region of a shock layer the flow is described with the set of Euler equations and adiabatic condition

$$\begin{gathered}(U - x)R_x + \tau R_\tau + RU_x = 0 \\ R[(U - x)U_x + \tau R_\tau] + P_x - 2\tau P/(1 + x\tau) = 0 \\ (U - x)(P_x/P - \gamma R_x/R) + \tau(P_\tau/P - \gamma R_\tau/R) + 2(\gamma - 1)\tau U/(1 + x\tau) = 0\end{gathered} \tag{4}$$

with Rankine-Hugoniot conditions at the shocks $x_j(\tau)$, $j = 1, 2$

$$\begin{gathered}U = \frac{2}{\gamma + 1}(x_k + \tau \dot{x}_k) + \frac{\gamma - 1}{\gamma + 1}\chi^{k-1} \\ R = \frac{\gamma + 1}{\gamma - 1}\lambda^{2(k-1)} \\ P = \frac{2}{\gamma + 1}\lambda^{2(k-1)}(\chi^{k-1} - (x_k + \tau \dot{x}_k))^2\end{gathered} \tag{5}$$

and non flowing and pressure equilibrium conditions at the contact surface $x_0(\tau)$

$$U = x_0 + \tau\dot{x}_0, \quad P = P_0 \equiv const \tag{6}$$

We will seek for the selfsimilar solutions of the problem (4)-(6) in two limiting cases : $\tau \ll 1$ and $\tau \gg 1$.

3. Planar solution

Let us consider the case $\tau \ll 1$ which corresponds to small times in the flow evolution. Then from (4) under assumption that functions depend only on x it follows

$$(U - x)R' + RU' = 0$$

$$(U - x)U' + R^{-1}P' = 0$$

$$(U - x)(P'/P - \gamma R'/R) = 0$$

The prime hereafter denotes a derivative with respect to x. These equations admit the solution with constant U, R, P, which are determined by the values at the shocks (5), while conditions (6) determine $x_{0,1,2}$

$$x_0 = \frac{1+\lambda\chi}{1+\lambda},\ x_1 = x_0 + \frac{\gamma-1}{2}\frac{\lambda(\chi-1)}{1+\lambda},\ x_2 = x_0 - \frac{\gamma-1}{2}\frac{\chi-1}{1+\lambda} \tag{7}$$

Let us call (7),(5) as a planar self-similar solution, since in essence it is a partial solution of the classic Riemann problem (Kotchine, 1926).

4. Radial solution

Let us consider the case $\tau \gg 1$ corresponding to large times in the flow evolution. From (4) under assumption that functions depend only on x it follows

$$\begin{aligned} &(U - x)R' + RU' = 0 \\ &(U - x)U' + R^{-1}(P' - 2P/x) = 0 \\ &(U - x)(P'/P - \gamma R'/R) + 2(\gamma - 1)U/x = 0 \end{aligned} \tag{8}$$

Note that system (8) with (5), (6) was considered earlier on other physical backgrounds by (Shidlovskaya, 1976) and (Chevalier & Imamura, 1983). These authors solved the problem numerically, by Runge-Kutta method. The exact analytic solution apparently does not exist since (8) may be reduced to a single Avel equation which is not integrable by quadratures. A comprehensive analysis of the problem is in the forthcoming paper, while in the present study we restrict ourself by consideration of the approximate analytic solution. We suppose that

$$(U-x)/x \ll 1 \tag{9}$$

Then (8) looks as

$$(U-x)R' + RU' = 0$$

$$P' - 2P/x = 0$$

$$\gamma U' + 2(\gamma - 1) = 0$$

so that it is easy solvable. The solution may be represented in the form

$$U = \frac{3\gamma - 2}{\gamma} x_0 - \frac{2(\gamma - 1)}{\gamma} x$$

$$R = R_0 \left| \frac{x - x_0}{x_0} \right|^{-\alpha}, \quad \alpha = \frac{2(\gamma - 1)}{3\gamma - 2} \tag{10}$$

$$P = P_0 \left(\frac{x}{x_0} \right)^2$$

where R_0 and P_0, as well as $x_{0,1,2}$, are determined from (5) and (6). It is convenient to present the coordinates of discontinuities in the form

$$x_0 = \frac{1 + \mu\chi}{1 + \mu}, \;\; x_1 = x_0 + f \frac{\mu(\chi - 1)}{1 + \mu}, \;\; x_2 = x_0 - f \frac{\chi - 1}{1 + \mu} \tag{11}$$

where

$$\mu = \lambda \left[1 + f \frac{(1 + \lambda)(\chi - 1)}{1 + \lambda\chi} \right], \quad f(\gamma) = \frac{\gamma(\gamma - 1)}{2(\gamma^2 + \gamma - 1)} \tag{12}$$

The solution (10)-(12) we will call as a radial self-similar solution.

The assumption (9) is fulfilled if

$$\xi_k \equiv |x_k - x_0|/x_0 \ll 1 \tag{13}$$

It is not difficult to show that the last inequality is fulfilled when χ is not so large and λ is close to 1.

The main result following after obtaining of the radial self-similar solution is: since x_0 (which also is the dimensionless velocity of contact surface) for the planar solution is less than for the radial self-similar one, it is conclusive that the contact surface is accelerated in time. This fact, as it is well-known (Chandrasekhar, 1961), can lead to the Rayleigh-Taylor instability for $\lambda < 1$ (density in the planar solution increases outwards) at least.

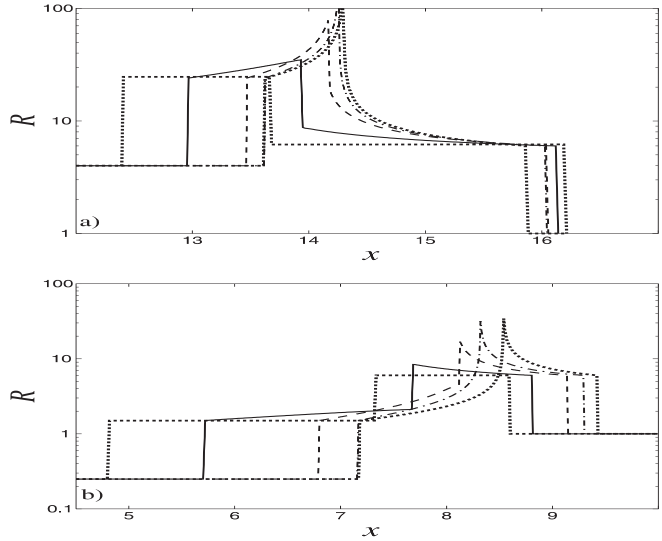

Figure 1. $R(x)$ at different moments of time for $\gamma = 1.4$, $\chi = 20$, $\lambda = 2$ (a) and $\lambda = 0.5$ (b). Points denote self-similar planar and radial solutions, solid, dashed and dash-dotted lines correspond to numerically obtained distributions for $\tau = 0.1, 0.7$ and 5 respectively.

5. Comparison with numerical solutions

To compare analytical solutions with numerical we run series of numerical simulations by making use of high resolution technique developed by Myasnikov (1996). We found that numerical solution behaves like the planar ones at first; very soon, however, the geometrical effects come into play, and as a result, the solution approaches to the radial self-similar one (Fig. 1). By the way, $\xi_{1,2} < 0.12$ and $\xi_{1,2} < 0.17$ for the cases presented at Fig .1,a and Fig. 1,b respectively. Note that relation (10) gives the infinite value of R at the contact discontinuity, while in numerical simulations R strives to a finite value at any fixed time when the grid resolution increases. More detailed analytical study, which is out of the scope of the present paper, confirmed the numerical results.

To explore the possible RT instability development, the planar analytical solution with the same parameters as presented at Fig. 1,b was introduced as an initial field for the 2D code in the sector with the angle range $-\pi/80 < \Theta < \pi/80$ at the moment $t = 0$. Then we consider a random perturbation with 10% amplitude with respect to u_1, which is applied to the velocity field of the gas between shocks. The preliminary results of simulations indicate that the flow is a subject of instability development, which manifests itself by developing 'fingers'

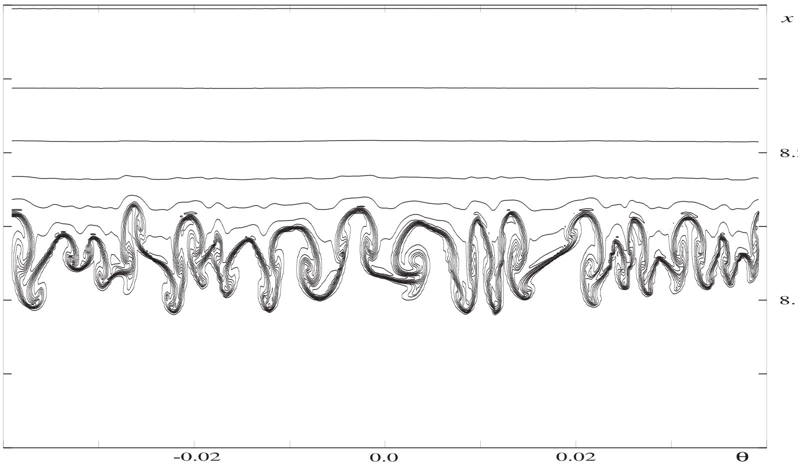

Figure 2. Logarithmic density contours in the vicinity of contact surface at $\tau = 0.7$ for the same set of parameters as in Fig. 1,b.

with mushroom-like caps (Fig. 2). At the same time, the additional study should be carried out to determine the dependence of the perturbed flow on the initial perturbation form, amplitude and numerical grid resolution.

A special attention should be also put on the case $\lambda > 1$, where the instability may be caused by the positive density gradient (Chandrasekhar, 1961), which appears at the beginning of the 1D flow evolution in the inner layer (Fig. 1,a).

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