DOI: https://doi.org/10.1070/IM8547

Many-circuit canard trajectories and their applications

S. D. Glyzin, A. Yu. Kolesov, and N. Kh. Rozov

Abstract. We study the case when two distinct curves of slow motion in a two-dimensional relaxation system with cylindrical phase space intersect each other in a generic way. We establish that the so-called canard trajectories can arise in this situation. They differ from ordinary canard trajectories in the following respect. The passage from the stable curve of slow motion to the unstable one is performed via finitely many asymptotically quick rotations of the phase point around the axis of the cylinder. The results obtained are used in the asymptotic analysis of eigenvalues of a boundary-value problem of Sturm–Liouville type for a singularly perturbed linear Schrödinger equation.

Keywords: singularly perturbed equation, many-circuit canard trajectories, asymptotics, boundary-value problems of Sturm–Liouville type.

§ 1. Statement of the problem and main result

The foundations of the asymptotic theory of relaxation oscillations in singularly perturbed systems were laid down by Pontryagin and Mishchenko (see, for example, [1]–[5] and the monograph [6]). It was shown in [7] that, under certain rather general assumptions, every system of ordinary differential equations with a small parameter in some derivatives is a C^1-perturbation of the corresponding relay system. Violation of these assumptions usually results in the appearance of canard trajectories: stable-unstable one-dimensional manifolds of slow motions.

Canard trajectories were first discovered in 1978 by the French mathematicians Francine Diener and Marc Diener using non-standard analysis. Then many authors studied canard trajectories using this method (see [8] and the references therein). This gave the impression that non-standard analysis is the most suitable tool for studying such trajectories. However, it was shown in [7] and elsewhere (see, for example, [9]–[11]) that canard trajectories in all possible situations admit a standard description using the techniques of classical asymptotic analysis.

In this paper we consider situations when a two-dimensional autonomous system of ordinary differential equations with a small parameter in one derivative has two distinct curves of slow motion which intersect each other generically. The existence of canard trajectories in such systems was first proved in [12], [13] using techniques

This paper was written with the support of the Russian Science Foundation (project no. 14-21-00158) and RFBR, grant no. 15-01-04066a.

AMS 2010 *Mathematics Subject Classification.* 34C26, 34E17, 34E20, 37G99.

of non-standard analysis. A standard version of these results is contained in [14]. Among the more recent papers on this theme, we mention [15]–[17].

We shall study the case of intersecting curves of slow motion in a two-dimensional system with cylindrical phase space and establish the existence of many-circuit canard trajectories in such systems.

The required relaxation system on the plane is assumed to be written as one scalar equation

$$\varepsilon\,\frac{d\theta}{dx} = f(x,\theta,\varepsilon,\mu), \tag{1.1}$$

where ε and μ are small parameters: $0<\varepsilon\ll 1$ and $|\mu|\ll 1$. Suppose that

$$f(x,\theta,\varepsilon,\mu)\in C^\infty([a,b]\times\mathbb{R}\times[0,\varepsilon_0]\times[-\mu_0,\mu_0]),\qquad f(x,\theta+T,\varepsilon,\mu)\equiv f(x,\theta,\varepsilon,\mu)$$

for some $a<b$, $\varepsilon_0,\mu_0>0$, $T>0$ and make the following assumptions.

Condition 1.1. *The equation $f(x,\theta,0,0)=0$ has exactly two solutions in the rectangle $\Pi_0=\{(x,\theta)\colon a\leqslant x\leqslant b,\ 0<\theta<T\}$:*

$$\theta=\varphi_j(x)\in C^\infty([a,b]),\qquad j=1,2. \tag{1.2}$$

These curves are naturally referred to as curves of slow motion for the equation (1.1) at $\varepsilon=0$, $\mu=0$.

Condition 1.2. *There is a point $x_0\in(a,b)$ such that*

$$\varphi_1(x_0)=\varphi_2(x_0),\qquad \varphi_1(x)-\varphi_2(x)<0\quad(>0)\quad\text{for}\quad x-x_0<0\quad(>0).$$

Moreover, the following inequalities hold on the half-open interval $a\leqslant x<x_0$:

$$\begin{gathered}(-1)^j f'_\theta(x,\varphi_j(x),0,0)>0,\qquad j=1,2,\\ f(x,\theta,0,0)<0\quad\text{for}\quad \varphi_1(x)<\theta<\varphi_2(x),\\ f(x,\theta,0,0)>0\quad\text{for}\quad 0\leqslant\theta<\varphi_1(x)\quad\text{and for}\quad \varphi_2(x)<\theta\leqslant T,\end{gathered}$$

and the following inequalities hold on the half-open interval $x_0<x\leqslant b$:

$$\begin{gathered}(-1)^j f'_\theta(x,\varphi_j(x),0,0)<0,\qquad j=1,2,\\ f(x,\theta,0,0)<0\quad\text{for}\quad \varphi_2(x)<\theta<\varphi_1(x),\\ f(x,\theta,0,0)>0\quad\text{for}\quad 0\leqslant\theta<\varphi_2(x)\quad\text{and for}\quad \varphi_1(x)<\theta\leqslant T.\end{gathered}$$

This condition means that the curves (1.2) when $\varepsilon=\mu=0$ are located as in Fig. 1. We notice that since f is periodic with respect to θ, there is actually a countable family of curves of slow motion. The pair of curves shown in Fig. 1 is repeated periodically in the rectangles

$$\Pi_n=\{(x,\theta)\colon a\leqslant x\leqslant b,\ nT<\theta<(n+1)T\},\qquad n\in\mathbb{Z}.$$

We further notice that the stable parts of these curves (the points where $f'_\theta<0$) are shown by solid lines, and the unstable parts (where $f'_\theta>0$) by broken lines. Clearly, at the point (x_0,θ_0), where $\theta_0=\varphi_1(x_0)=\varphi_2(x_0)$, we have $f'_\theta(x_0,\theta_0,0,0)=0$.

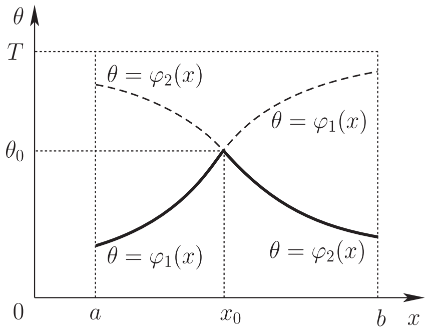

Figure 1

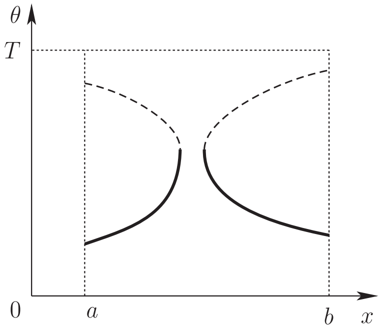

Figure 2. $\mu > 0$

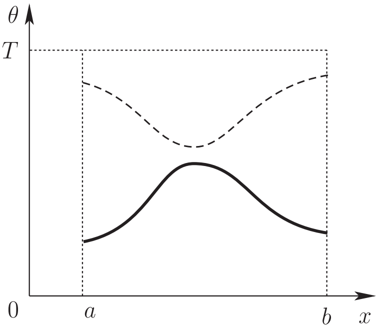

Figure 3. $\mu < 0$

Condition 1.3. *We assume that*

$$\varphi_1'(x_0) - \varphi_2'(x_0) > 0, \qquad f_\mu'(x_0, \theta_0, 0, 0) > 0, \qquad f_{\theta\theta}''(x_0, \theta_0, 0, 0) > 0. \tag{1.3}$$

Conditions 1.1 and 1.2 along with the first inequality in (1.3) guarantee that the intersection of the slow-motion curves (1.2) at the point (x_0, θ_0) is transversal. The auxiliary parameter μ characterizes the 'deviation' from this singularity, which occurs in a generic way because of the second inequality in (1.3). The third inequality in (1.3) is also a genericity condition. Finally, Conditions 1.1–1.3 together guarantee that the slow-motion curves, which are given by the equation $f(x, \theta, 0, \mu) = 0$, evolve with respect to μ as shown in Figs. 2, 3.

We stress that all these conditions can be verified by local methods. Indeed, suppose that the following relations hold at some point (x_0, θ_0), where $a < x_0 < b$, $0 < \theta_0 < T$:

$$f(x_0, \theta_0, 0, 0) = f_x'(x_0, \theta_0, 0, 0) = f_\theta'(x_0, \theta_0, 0, 0) = 0, \qquad f_\mu'(x_0, \theta_0, 0, 0) \neq 0. \tag{1.4}$$

Consider the polynomial

$$\begin{gathered} P(\lambda) = p_1\lambda^2 + p_2\lambda + p_3, \\ p_1 = f_{\theta\theta}''(x_0, \theta_0, 0, 0), \qquad p_2 = 2f_{x\theta}''(x_0, \theta_0, 0, 0), \qquad p_3 = f_{xx}''(x_0, \theta_0, 0, 0) \end{gathered} \tag{1.5}$$

and assume that it has two real roots $\lambda_1 > \lambda_2$. Then the standard theory of branching [18] guarantees that the equation

$$f(x, \theta, 0, 0) = 0$$

in a sufficiently small neighbourhood of $x = x_0$ has two solutions $\theta = \varphi_j(x)$, $j = 1, 2$, such that $\varphi_1(x_0) = \varphi_2(x_0) = \theta_0$, $\varphi_1'(x_0) = \lambda_1$, $\varphi_2'(x_0) = \lambda_2$. Therefore, choosing the numbers a, b sufficiently close to x_0 and replacing θ by $-\theta$ and μ by $-\mu$ in (1.1) if necessary, we see that Conditions 1.1–1.3 hold. Conversely, Conditions 1.1–1.3 automatically guarantee that (1.4) holds and the polynomial (1.5) has two distinct roots $\lambda_1 = \varphi_1'(x_0)$, $\lambda_2 = \varphi_2'(x_0)$.

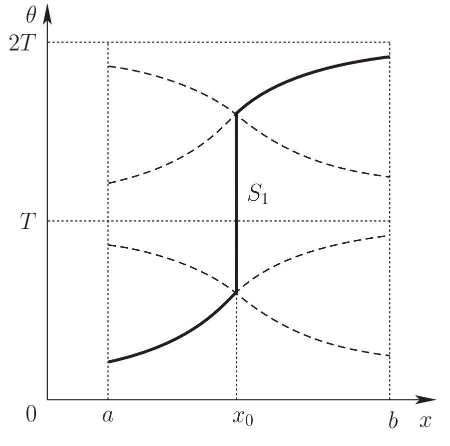

Figure 4

It follows from Condition 1.2 that the curve

$$S_0 = \{(x, \theta)\colon a \leqslant x \leqslant b,\ \theta = \varphi_1(x)\} \tag{1.6}$$

splits into a stable part (for $x < x_0$) and an unstable part (for $x > x_0$). Therefore it is a canard trajectory of the degenerate equation $f(x, \theta, 0, 0) = 0$. We call (1.6) a zero-circuit canard trajectory. In general, n-circuit canard trajectories are defined by the equalities

$$\begin{aligned} S_n = \{(x, \theta)\colon a \leqslant x \leqslant x_0,\ \theta = \varphi_1(x)\} \cup \{(x, \theta)\colon x = x_0,\ \theta_0 \leqslant \theta \leqslant \theta_0 + nT\} \\ \cup \{(x, \theta)\colon x_0 \leqslant x \leqslant b,\ \theta = \varphi_1(x) + nT\}, \qquad n \in \mathbb{N}. \end{aligned} \tag{1.7}$$

These trajectories are characterized by the presence of a vertical interval

$$\{(x, \theta)\colon x = x_0,\ \theta_0 \leqslant \theta \leqslant \theta_0 + nT\}$$

which is passed instantaneously as x varies. Geometrically, this interval corresponds to n full circuits of the point (x, θ) around the axis of the cylinder

$$C = \{(x, \theta)\colon a \leqslant x \leqslant b,\ 0 \leqslant \theta \leqslant T \pmod{T}\}. \tag{1.8}$$

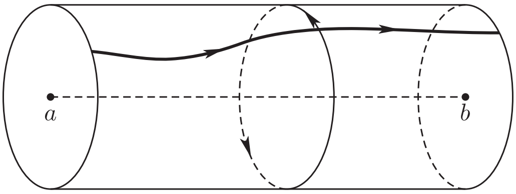

Figure 5

A visual impression of the trajectories (1.7) is given by Figs. 4 and 5 in the case when $n = 1$.

By a canard trajectory of the original equation (1.1) we mean any solution whose zero approximation as $\varepsilon \to 0$ is the part of the curve (1.6) or (1.7) with $x_1 \leqslant x \leqslant x_2$, where $x_1 \in [a, x_0)$ and $x_2 \in (x_0, b]$. In particular, a canard trajectory whose zero approximation includes the curve (1.7), is a solution (if it exists) of the boundary-value problem for (1.1) with boundary conditions

$$\theta\big|_{x=a} = \theta_1, \qquad \theta\big|_{x=b} = \theta_2 + nT, \tag{1.9}$$

where

$$0 \leqslant \theta_1 < \varphi_2(a), \qquad \varphi_2(b) < \theta_2 \leqslant T. \tag{1.10}$$

The boundary-value problem (1.1), (1.9) is a convenient object of study because it enables us to distinguish a unique n-circuit canard trajectory, thus giving a certain strategy for 'duck hunting'.

Before stating our main result, we enhance somewhat the curve (1.7). Namely, consider the curve $\Gamma_n = S_n \cup \Sigma_1 \cup \Sigma_2$, where Σ_1 is the closed vertical interval connecting the points (a, θ_1) and $(a, \varphi_1(a))$, and Σ_2 is the analogous interval with endpoints $(b, \theta_2 + nT)$ and $(b, \varphi_1(b) + nT)$. We also put

$$\mu_n = \frac{(2n+1)(\varphi_1'(x_0) - \varphi_2'(x_0)) + \varphi_1'(x_0) + \varphi_2'(x_0) - 2f_\varepsilon'(x_0, \theta_0, 0, 0)}{2f_\mu'(x_0, \theta_0, 0, 0)}. \tag{1.11}$$

Theorem 1.1. *Suppose that Conditions 1.1–1.3 and inequalities (1.10) hold. Fix an arbitrary integer $n \geqslant 0$. Then for all sufficiently small $\varepsilon > 0$ there is a unique value $\mu_n(\varepsilon)$, $\mu_n(0) = 0$, of the parameter μ such that the boundary-value problem (1.1), (1.9) has a solution $\theta = \theta_n(x, \varepsilon)$. Moreover, the curves*

$$\Gamma_n(\varepsilon) = \{(x, \theta) \colon a \leqslant x \leqslant b,\ \theta = \theta_n(x, \varepsilon)\} \tag{1.12}$$

and the functions $\mu_n(\varepsilon)$ satisfy the limiting relations

$$\lim_{\varepsilon \to 0} \Gamma_n(\varepsilon) = \Gamma_n, \qquad \lim_{\varepsilon \to 0} \frac{\mu_n(\varepsilon)}{\varepsilon} = \mu_n. \tag{1.13}$$

This theorem was proved for $n = 0$ in [14]. A proof for $n \geqslant 1$ will be given in the next section.

§ 2. Proof of Theorem 1.1

2.1. Auxiliary constructions. A value $\mu = \mu_n(\varepsilon)$, $\mu_n(0) = 0$, of the parameter μ such that the boundary-value problem (1.1), (1.9) is soluble, must exist for general qualitative reasons. To see this, we fix a sufficiently small $\mu \neq 0$ and consider the solutions

$$\theta = \theta_j(x,\varepsilon,\mu), \quad j = 1,2; \qquad \theta_1(a,\varepsilon,\mu) = \theta_1, \quad \theta_2(b,\varepsilon,\mu) = \theta_2 + nT \tag{2.1}$$

of the equation (1.1) for $0 < \varepsilon \ll 1$.

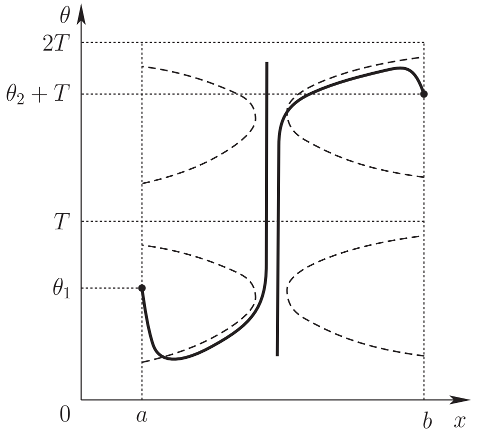

Figure 6. $\mu > 0$

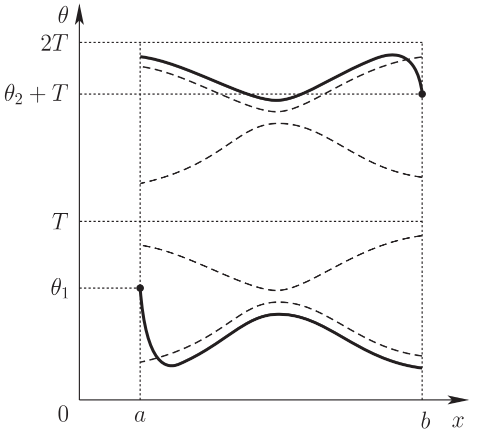

Figure 7. $\mu < 0$

First suppose that $\mu > 0$. Then we conclude from (1.10) in view of the location of the slow-motion curves (see Fig. 2) that the solutions (2.1) are of the form shown in Fig. 6 (where we have put $n = 1$). Indeed, by the results in [19], the first solution behaves as follows. As x increases, the solution first 'falls' onto the stable 'slow' curve, that is, it moves asymptotically quickly in an asymptotically small neighbourhood of the ray $\theta = \theta_1$ from the point (a, θ_1) up to its intersection with the curve mentioned. Then the motion occurs in the ε-neighbourhood of the 'slow' curve, and then we observe an abrupt exit (as described in [6], [7]) followed by a quick motion along some vertical ray up to asymptotically large values of θ. The situation for the second solution (2.1) is symmetric (see Fig. 6).

When $\mu < 0$, the solutions (2.1) take the form shown (for $n = 1$) in Fig. 7. This also follows from (1.10) in view of the mutual position of the slow-motion curves (see Fig. 3). Using these facts and a simple continuity argument, we conclude that there is at least one value $\mu = \mu_n(\varepsilon)$, $\mu_n(0) = 0$, such that the solutions (2.1) coincide and, moreover, the corresponding curve (1.12) satisfies the first limiting equality in (1.13). Thus, we only need to prove that the value $\mu_n(\varepsilon)$ is unique and the second limiting equality in (1.13) holds.

Our description of the qualitative behaviour of the solutions (2.1) shows that, as x increases or decreases, they continue into any fixed sufficiently small fixed neighbourhoods of the points (x_0, θ_0), $(x_0, \theta_0 + nT)$ respectively. These points can be regarded as coinciding since the equation (1.1) is periodic. Therefore our first task is to study this equation near the singular point (x_0, θ_0).

For convenience, we write the equation (1.1) in so-called normal form near (x_0, θ_0). This is done by making the change of variables

$$\theta - \varphi_1(x) = \varkappa(\eta - \xi), \quad \theta - \varphi_2(x) = \varkappa(\eta + \xi), \qquad \varkappa = \sqrt{\frac{\varphi_1'(x_0) - \varphi_2'(x_0)}{f''_{\theta\theta}(x_0, \theta_0, 0, 0)}}, \tag{2.2}$$

which transforms the curves (1.2) into the lines $\eta = \pm\xi$. Thus we obtain the singularly perturbed equation

$$\varepsilon \frac{d\eta}{d\xi} = \gamma(\xi, \eta)(\eta^2 - \xi^2) + \mu\Delta_1(\xi, \eta, \varepsilon, \mu) + \varepsilon\Delta_2(\xi, \eta, \varepsilon, \mu), \tag{2.3}$$

where the functions γ and Δ_j, $j = 1, 2$, are smooth in a sufficiently small neighbourhood of the origin and satisfy

$$\begin{gathered} \gamma(0,0) = 1, \qquad \Delta_1(0,0,0,0) = \frac{2f'_\mu(x_0, \theta_0, 0, 0)}{\varphi_1'(x_0) - \varphi_2'(x_0)}, \\ \Delta_2(0,0,0,0) = \frac{2f'_\varepsilon(x_0, \theta_0, 0, 0) - \varphi_1'(x_0) - \varphi_2'(x_0)}{\varphi_1'(x_0) - \varphi_2'(x_0)}. \end{gathered} \tag{2.4}$$

The equation (2.3) is called the normal form of (1.1). We shall study it for $|\xi| \leqslant q$ and $|\eta| \leqslant r$, where $q, r > 0$ are sufficiently small.

Put

$$\mu = \varepsilon\mu_n + \varepsilon^{3/2}\delta \tag{2.5}$$

in (2.3), where $\delta \in \mathbb{R}$ is a parameter of order 1 (varying over a fixed compact set Ω), which will later be used to 'match' the solutions (2.1). Then the normal form can be written as

$$\varepsilon \frac{d\eta}{d\xi} = \gamma(\xi, \eta)(\eta^2 - \xi^2) + \varepsilon\Delta(\xi, \eta, \varepsilon, \sqrt{\varepsilon}\,\delta), \tag{2.6}$$

where the remainder term

$$\Delta(\xi, \eta, \varepsilon, \nu) = \frac{1}{\varepsilon}\big(\mu\Delta_1(\xi, \eta, \varepsilon, \mu) + \varepsilon\Delta_2(\xi, \eta, \varepsilon, \mu)\big)\big|_{\mu = \varepsilon\mu_n + \varepsilon\nu} \tag{2.7}$$

possesses the following property because of (1.11), (2.4):

$$\Delta(0,0,0,0) = 2n + 1. \tag{2.8}$$

2.2. Main lemmas. We write

$$\eta = \mathcal{F}_j(\xi, \varepsilon, \delta), \qquad j = 1, 2, \tag{2.9}$$

for the solutions of (2.6) that correspond to the functions

$$\theta_1(x, \varepsilon, \mu)\big|_{\mu = \varepsilon\mu_n + \varepsilon^{3/2}\delta}, \qquad (\theta_2(x, \varepsilon, \mu) - nT)\big|_{\mu = \varepsilon\mu_n + \varepsilon^{3/2}\delta}.$$

By the results in [19], they are defined for $-q \leqslant \xi \leqslant -q_0$ and $q_0 \leqslant \xi \leqslant q$ respectively, where $q_0 > 0$ is an arbitrary fixed sufficiently small number. Moreover, the following asymptotic representations for the solutions (2.9) as $\varepsilon \to 0$ hold uniformly with respect to ξ in these intervals and with respect to $\delta \in \Omega$:

$$\mathcal{F}_j(\xi, \varepsilon, \delta) = \xi + \varepsilon a_1(\xi, \delta) + \varepsilon^{3/2} a_2(\xi, \delta) + O(\varepsilon^2), \qquad j = 1, 2, \tag{2.10}$$

where

$$a_1(\xi, \delta) = \frac{1 - \Delta(\xi, \xi, 0, 0)}{2\xi\gamma(\xi, \xi)}, \qquad a_2(\xi, \delta) = -\frac{\delta\Delta_1(\xi, \xi, 0, 0)}{2\xi\gamma(\xi, \xi)}. \tag{2.11}$$

We also notice that these representations remain valid after differentiation with respect to δ.

The formulae (2.10) certainly do not hold as $\xi \to 0$ since in this case (see (2.4), (2.7), (2.8), (2.11)) we have

$$a_1(\xi, \delta) = -\frac{n}{\xi} + \varkappa_1 + O(\xi), \qquad a_2(\xi, \delta) = \frac{\varkappa_2}{\xi} + O(1), \tag{2.12}$$

where

$$\begin{gathered} \varkappa_1 = -\frac{1}{2}\big(\Delta'_\xi(0, 0, 0, 0) + \Delta'_\eta(0, 0, 0, 0)\big) + n\big(\gamma'_\xi(0, 0) + \gamma'_\eta(0, 0)\big), \\ \varkappa_2 = -\frac{1}{2}\delta\Delta_1(0, 0, 0, 0). \end{gathered} \tag{2.13}$$

However, they can be 'adjusted' for $-q \leqslant \xi \leqslant -\varepsilon^{\lambda_1}$ and $\varepsilon^{\lambda_1} \leqslant \xi \leqslant q$, where $\lambda_1 = \mathrm{const} \in (0, 1/2)$.

Lemma 2.1. *As $\varepsilon \to 0$, the following asymptotic formulae hold uniformly for $-q \leqslant \xi \leqslant -\varepsilon^{\lambda_1}$ (resp. $\varepsilon^{\lambda_1} \leqslant \xi \leqslant q$) and $\delta \in \Omega$:*

$$\mathcal{F}_j(\xi, \varepsilon, \delta) = \xi + \varepsilon a_1(\xi, \delta) + \varepsilon^{3/2} a_2(\xi, \delta) + O(\varepsilon^{2-3\lambda_1}), \qquad j = 1, 2. \tag{2.14}$$

Proof. We shall establish the asymptotic formula (2.14) for $j = 1$ (the case $j = 2$ is treated similarly). To do this, we substitute the expression $\eta = \xi + \varepsilon a_1(\xi, \delta) + \varepsilon^{3/2} a_2(\xi, \delta) + z$ into (2.6). This yields the following equation for z:

$$\varepsilon \frac{dz}{d\xi} = h(\xi, \varepsilon, \delta) z + \Phi(z, \xi, \varepsilon, \delta) + \varepsilon^2 \Psi(\xi, \varepsilon, \delta), \tag{2.15}$$

where the functions h, Φ, Ψ satisfy the conditions

$$h(\xi, \varepsilon, \delta) = h_0(\xi) + O(\varepsilon/\xi), \quad \varepsilon \to 0; \qquad h_0(\xi) = 2\xi + O(\xi^2), \quad \xi \to 0, \tag{2.16}$$

$$\Phi(0, \xi, \varepsilon, \delta) \equiv \frac{\partial\Phi}{\partial z}(0, \xi, \varepsilon, \delta) \equiv 0, \qquad \left|\frac{\partial\Phi}{\partial z}(z, \xi, \varepsilon, \delta)\right| \leqslant M|z| \quad \text{for} \quad |z| \leqslant 1, \tag{2.17}$$

$$|\Psi(\xi, \varepsilon, \delta)| \leqslant \frac{M}{\xi^2}. \tag{2.18}$$

Here and in what follows we write M for various universal positive constants (independent of ξ, ε and so on) whose exact values are irrelevant.

We pass from the equation (2.15) to the integral equation

$$z(\xi,\varepsilon,\delta) = z(-q,\varepsilon,\delta)\exp\left[\frac{1}{\varepsilon}\int_{-q}^{\xi} h(\sigma,\varepsilon,\delta)\,d\sigma\right] + \frac{1}{\varepsilon}\int_{-q}^{\xi}\exp\left[\frac{1}{\varepsilon}\int_{\sigma}^{\xi} h(\sigma',\varepsilon,\delta)\,d\sigma'\right]\left[\Phi(z,\sigma,\varepsilon,\delta)+\varepsilon^2\Psi(\sigma,\varepsilon,\delta)\right]d\sigma, \tag{2.19}$$

where (see (2.10))

$$z(-q,\varepsilon,\delta) = \mathcal{F}_1(-q,\varepsilon,\delta) + q - \varepsilon a_1(-q,\delta) - \varepsilon^{3/2}a_2(-q,\delta) = O(\varepsilon^2). \tag{2.20}$$

Using the properties (2.16)–(2.18) and (2.20), the requirement $\lambda_1 < 1/2$ and the inequality

$$\frac{1}{\varepsilon}\int_{-q}^{\xi}\exp\left[\frac{1}{\varepsilon}\int_{\sigma}^{\xi} h(\sigma',\varepsilon,\delta)\,d\sigma'\right]d\sigma \leqslant \frac{M}{|\xi|},$$

which follows from (2.16), we easily see that the operator generated by the right-hand side of (2.19) maps some ball with centre zero and radius of order $\varepsilon^{2-3\lambda_1}$ in the space $C([-q,-\varepsilon^{\lambda_1}])$ to itself and this map is a contraction (with constant of contraction of order $\varepsilon^{2-4\lambda_1}$). Hence there is a unique function $z(\xi,\varepsilon,\delta)$ which is smooth with respect to all variables, belongs to the ball mentioned, and is a solution of the desired equation. □

In addition to the lemma just proved, we notice that the formulae (2.14) can be differentiated with respect to δ. This is verified by an analysis of the linear integral equation for $\partial z/\partial\delta$, which is similar to (2.19). The calculations are easy and we omit them.

At the next step we perform the changes $\eta = \sqrt{\varepsilon}\,v$, $\xi = \sqrt{\varepsilon}\,\tau$ in the equation (2.6), bringing it into the form

$$\frac{dv}{d\tau} = \gamma(\sqrt{\varepsilon}\,\tau,\sqrt{\varepsilon}\,v)(v^2-\tau^2) + \Delta(\sqrt{\varepsilon}\,\tau,\sqrt{\varepsilon}\,v,\varepsilon,\sqrt{\varepsilon}\,\delta). \tag{2.21}$$

Putting $\varepsilon = 0$ in (2.21) and using (2.4), (2.8), we arrive at the model equation

$$\frac{dv}{d\tau} = v^2 - \tau^2 + 2n + 1. \tag{2.22}$$

We discuss separately how to choose the particular solutions of (2.22) that correspond to the canard trajectories. To do this, we consider the more general equation

$$\frac{dv}{d\tau} = v^2 - \tau^2 + \alpha \tag{2.23}$$

with a parameter $\alpha \in \mathbb{R}$. By [14], it has two special solutions $v = v_-(\tau,\alpha)$ and $v = v_+(\tau,\alpha)$ defined on the intervals $-\infty < \tau \leqslant -\overline{\tau}$ and $\overline{\tau} \leqslant \tau < +\infty$ respectively, where $\overline{\tau} > 0$ is sufficiently large, and possess the following properties:

$$\begin{gathered} v_-(\tau,\alpha) = \tau + \sum_{k=1}^{\infty}\frac{\beta_{2k-1}}{\tau^{2k-1}} \quad \text{as} \quad \tau \to -\infty, \qquad \beta_1 = -\frac{\alpha-1}{2}, \\ v_+(\tau,\alpha) = -v_-(-\tau,\alpha) \end{gathered} \tag{2.24}$$

(we do not give explicit formulae for β_{2k-1} with $k \geqslant 2$ since they are not used in what follows). The relations (2.24) show that these solutions are analogous to the stable manifold $\eta = \xi$, $\xi < 0$, and the unstable manifold $\eta = \xi$, $\xi > 0$, of slow motions for the equation (2.6).

We say that the solutions $v = v_\pm(\tau, \alpha)$ of (2.23) are conjugate for some α if there is a solution $u(\tau, \alpha) \not\equiv 0$ of the auxiliary linear equation

$$\frac{d^2u}{d\tau^2} + (\alpha - \tau^2)u = 0 \tag{2.25}$$

such that

$$v_-(\tau, \alpha) = v_+(\tau, \alpha) = v_0(\tau, \alpha) \stackrel{\text{def}}{=} -\frac{1}{u(\tau, \alpha)} \frac{d}{d\tau} u(\tau, \alpha). \tag{2.26}$$

The equalities (2.26) mean that the solution $v_-(\tau, \alpha)$ extends to the semi-axis $\tau \geqslant -\overline{\tau}$ in such a way that this extension has finitely many singularities of pole type (they correspond to the zeros of the solution $u(\tau, \alpha)$ of (2.25)) on this semi-axis and, most importantly, coincides with the solution $v_+(\tau, \alpha)$ for $\tau \geqslant \overline{\tau}$. Hence we can assert that the function $v_0(\tau, \alpha)$ in (2.26) is an analogue of a many-circuit canard trajectory (the circuits correspond to the poles of this function).

Thus, it becomes clear that we are interested in those values of α where conjugacy occurs. It follows from (2.24), (2.26) and the formula $v = -u^{-1} du/d\tau$, which relates any pair of solutions of the equations (2.23) and (2.25), that these values coincide with the spectrum of the boundary-value problem

$$\frac{d^2u}{d\tau^2} + (\alpha - \tau^2)u = 0, \qquad u \to 0 \quad \text{as} \quad \tau \to \pm\infty.$$

It is known [20] that this problem has eigenvalues $\alpha_n = 2n + 1$, $n = 0, 1 \ldots$, and the corresponding eigenfunctions are

$$u_n = \exp\left(-\frac{\tau^2}{2}\right) H_n(\tau), \qquad H_n(\tau) = (-1)^n \exp(\tau^2) \frac{d^n}{d\tau^n}\bigl(\exp(-\tau^2)\bigr). \tag{2.27}$$

We also notice that the $H_n(\tau)$ are the well-known Chebyshev–Hermite polynomials having exactly n zeros $\tau_1 < \tau_2 < \cdots < \tau_n$ on the whole axis $\tau \in \mathbb{R}$.

We now reconsider the equation (2.22) and, in accordance with the discussion above, choose the particular solution

$$v_0(\tau) = -\frac{1}{u_n(\tau)} \frac{d}{d\tau} u_n(\tau), \qquad v_0(-\tau) = -v_0(\tau) \tag{2.28}$$

of this equation, where u_n is the function in (2.27). This solution has exactly n poles at the points $\tau_1 < \tau_2 < \cdots < \tau_n$. The graph of the function (2.28) for $n = 2$ is sketched in Fig. 8.

Let $v_j = v_j(\tau, \varepsilon, \delta)$, $j = 1, 2$, be the solutions of (2.21) with initial conditions

$$v_1\big|_{\tau = -\varepsilon^{\lambda_1 - 1/2}} = \varepsilon^{-1/2} \mathcal{F}_1(-\varepsilon^{\lambda_1}, \varepsilon, \delta), \qquad v_2\big|_{\tau = \varepsilon^{\lambda_1 - 1/2}} = \varepsilon^{-1/2} \mathcal{F}_2(\varepsilon^{\lambda_1}, \varepsilon, \delta), \tag{2.29}$$

which are borrowed from the previous step. We arbitrarily fix some values $\tau_* \in (-\infty, \tau_1)$, $\tau_{**} \in (\tau_n, +\infty)$ and seek the asymptotic behaviour of the solutions

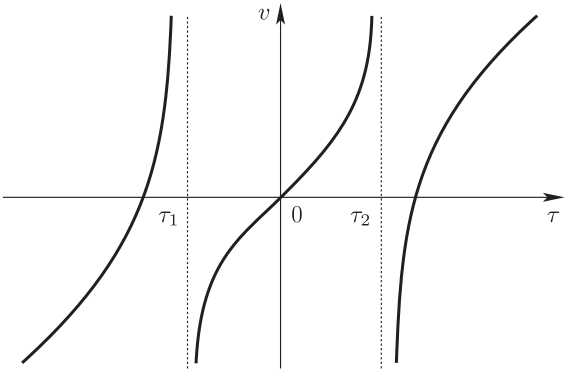

Figure 8

$v_j(\tau,\varepsilon,\delta)$, $j=1,2$, on the intervals $-\varepsilon^{\lambda_1-1/2}\leqslant\tau\leqslant\tau_*$ and $\tau_{**}\leqslant\tau\leqslant\varepsilon^{\lambda_1-1/2}$ respectively in the form

$$v_j=v_0(\tau)+\sqrt{\varepsilon}\,v_{1,j}(\tau,\delta)+\cdots,\qquad j=1,2. \tag{2.30}$$

Substituting (2.30) into (2.21) and equating the coefficients of $\sqrt{\varepsilon}$, we arrive at the following linear inhomogeneous equations for the functions $v_{1,j}$:

$$\frac{dv_{1,j}}{d\tau}=2v_0(\tau)v_{1,j}+g(\tau), \tag{2.31}$$

where

$$\begin{aligned}g(\tau)=\bigl(v_0^2(\tau)-\tau^2\bigr)\bigl(\tau\gamma'_\xi(0,0)+v_0(\tau)\gamma'_\eta(0,0)\bigr)+\tau\Delta'_\xi(0,0,0,0)\\ +v_0(\tau)\Delta'_\eta(0,0,0,0)+\delta\Delta_1(0,0,0,0).\end{aligned} \tag{2.32}$$

When $j=1$, we choose $v_{1,1}$ to be the only solution of (2.31) which is bounded as $\tau\to-\infty$. This solution is determined by the equality

$$v_{1,1}(\tau,\delta)=\int_{-\infty}^{\tau}\exp\biggl[\int_\sigma^\tau 2v_0(\sigma')\,d\sigma'\biggr]g(\sigma)\,d\sigma. \tag{2.33}$$

Indeed, using the asymptotic representation for $v_0(\tau)$ (see (2.24) with $\alpha=2n+1$), we see from (2.32), (2.33) that

$$v_{1,1}(\tau,\delta)=\varkappa_1+O\Bigl(\frac{1}{\tau}\Bigr),\qquad \tau\to-\infty, \tag{2.34}$$

where $\varkappa_1$ is the constant in (2.13). When $j=2$, we similarly have

$$\begin{gathered}v_{1,2}(\tau,\delta)=-\int_\tau^{+\infty}\exp\biggl[\int_\sigma^\tau 2v_0(\sigma')\,d\sigma'\biggr]g(\sigma)\,d\sigma,\\ v_{1,2}(\tau,\delta)=\varkappa_1+O\Bigl(\frac{1}{\tau}\Bigr),\qquad \tau\to+\infty.\end{gathered} \tag{2.35}$$

The following lemma gives a rigorous meaning to these constructions.

Lemma 2.2. *Suppose that the parameter* λ_1 *belongs to the interval* $(1/3, 1/2)$. *Then the following asymptotic representations as* $\varepsilon \to 0$ *hold on the closed intervals* $\tau \in [-\varepsilon^{\lambda_1-1/2}, \tau_*]$ *and* $\tau \in [\tau_{**}, \varepsilon^{\lambda_1-1/2}]$ *respectively*:

$$\begin{gathered} v_j(\tau,\varepsilon,\delta) = v_0(\tau) + \sqrt{\varepsilon}\, v_{1,j}(\tau,\delta) + \varepsilon^{3/2-3\lambda_1} R_j(\tau,\varepsilon,\delta), \\ |R_j| + \left|\frac{\partial R_j}{\partial \delta}\right| \leqslant M\biggl(\frac{u_n((-1)^j\varepsilon^{\lambda_1-1/2})}{u_n(\tau)}\biggr)^2 + \varepsilon^{3\lambda_1-1/2} M(|\tau|+1), \qquad j=1,2. \end{gathered} \tag{2.36}$$

Proof. As in the proof of Lemma 2.1, we consider only the case $j=1$. Substituting the expression $v = v_0 + \sqrt{\varepsilon} v_{1,1} + z$ into (2.21), we obtain the following equation for z (an analogue of (2.15)):

$$\frac{dz}{d\tau} = h(\tau,\varepsilon,\delta)z + \Phi(z,\tau,\varepsilon,\delta) + \varepsilon\Psi(\tau,\varepsilon,\delta), \tag{2.37}$$

where the functions h, Φ, Ψ satisfy the conditions

$$\begin{gathered} h(\tau,\varepsilon,\delta) = 2v_0(\tau) + O\bigl(\sqrt{\varepsilon}(\tau^2+1)\bigr), \qquad \Phi(0,\tau,\varepsilon,\delta) \equiv \frac{\partial\Phi}{\partial z}(0,\tau,\varepsilon,\delta) \equiv 0, \\ \left|\frac{\partial\Phi}{\partial z}(z,\tau,\varepsilon,\delta)\right| \leqslant M|z| \quad \text{for} \quad |z| \leqslant 1, \qquad |\Psi(\tau,\varepsilon,\delta)| \leqslant M(|\tau|+1)^2 \end{gathered} \tag{2.38}$$

uniformly with respect to $\tau \in [-\varepsilon^{\lambda_1-1/2}, \tau_*]$ and $\delta \in \Omega$. By (2.29), the equation (2.37) must be endowed with the initial condition

$$z(-\varepsilon^{\lambda_1-1/2},\varepsilon,\delta) = \varepsilon^{-1/2}\bigl[\mathcal{F}_1(-\varepsilon^{\lambda_1},\varepsilon,\delta) - \sqrt{\varepsilon}\, v_0(-\varepsilon^{\lambda_1-1/2}) - \varepsilon v_{1,1}(-\varepsilon^{\lambda_1-1/2},\delta)\bigr]. \tag{2.39}$$

Using the representations (2.12), (2.14), (2.24), (2.34) and the requirement $\lambda_1 \in (1/3, 1/2)$, we see from (2.39) that

$$z(-\varepsilon^{\lambda_1-1/2},\varepsilon,\delta) = O(\varepsilon^{3/2-3\lambda_1}), \qquad \frac{\partial z}{\partial\delta}(-\varepsilon^{\lambda_1-1/2},\varepsilon,\delta) = O(\varepsilon^{3/2-3\lambda_1}). \tag{2.40}$$

Our further analysis proceeds by passing from the Cauchy problem (2.37), (2.39) to the corresponding integral equation

$$\begin{aligned} z(\tau,\varepsilon,\delta) &= z(-\varepsilon^{\lambda_1-1/2},\varepsilon,\delta)\exp\biggl[\int_{-\varepsilon^{\lambda_1-1/2}}^{\tau} h(\sigma,\varepsilon,\delta)\,d\sigma\biggr] \\ &\quad + \int_{-\varepsilon^{\lambda_1-1/2}}^{\tau} \exp\biggl[\int_{\sigma}^{\tau} h(\sigma',\varepsilon,\delta)\,d\sigma'\biggr]\bigl[\Phi(z,\sigma,\varepsilon,\delta) + \varepsilon\Psi(\sigma,\varepsilon,\delta)\bigr]\,d\sigma. \end{aligned} \tag{2.41}$$

Using (2.38), (2.40) and the bounds

$$\begin{gathered} \int_{-\varepsilon^{\lambda_1-1/2}}^{\tau} \exp\biggl[\int_{\sigma}^{\tau} h(\sigma',\varepsilon,\delta)\,d\sigma'\biggr](|\sigma|+1)^k\,d\sigma \leqslant M(|\tau|+1)^{k-1}, \qquad k=0,1,2, \\ \exp\biggl[\int_{\sigma}^{\tau} h(\sigma',\varepsilon,\delta)\,d\sigma'\biggr] \leqslant M\biggl(\frac{u_n(\sigma)}{u_n(\tau)}\biggr)^2, \qquad \sigma \leqslant \tau, \end{gathered}$$

which follow from the first equality (2.38), we see that the operator generated by the right-hand side of (2.41) maps a certain set $K \subset C([-\varepsilon^{\lambda_1-1/2}, \tau_*])$ of the form

$$K = \left\{ z(\tau) \colon |z(\tau)| \leqslant M\varepsilon^{3/2-3\lambda_1} \left(\frac{u_n\left(-\varepsilon^{\lambda_1-1/2}\right)}{u_n(\tau)} \right)^2 + \varepsilon M(|\tau|+1) \right\}$$

to itself and this map is a contraction (with constant of contraction of order $\varepsilon^{2-4\lambda_1}$). Thus the equation (2.41) has a unique solution $z(\tau, \varepsilon, \delta) \in K$. It is also easy to prove that $\partial z/\partial\delta \in K$. $\square$

Here is a plan of what follows. Take the solution $v_1(\tau, \varepsilon, \delta)$ and extend it (in several steps) with respect to τ beyond the point $\tau = \tau_n$. More precisely, the function $v_1(\tau, \varepsilon, \delta)$ will be defined only on intervals of the form

$$[\tau_*, \tau_1 - \varepsilon^{\lambda_2}], \qquad [\tau_k + \varepsilon^{\lambda_2}, \tau_{k+1} - \varepsilon^{\lambda_2}], \quad k = 1, \dots, n-1, \qquad [\tau_n + \varepsilon^{\lambda_2}, \tau_{**}],$$

where τ_k are the zeros of the polynomial $H_n(\tau)$ and $\lambda_2 > 0$ is a constant. When we pass from $\tau = \tau_k - \varepsilon^{\lambda_2}$ to $\tau = \tau_k + \varepsilon^{\lambda_2}$, the argument becomes non-local: the phase point $(x, \theta_1(x, \varepsilon, \mu))\big|_{\mu=\varepsilon\mu_n+\varepsilon^{3/2}\delta}$ leaves the neighbourhood of (x_0, θ_0), makes a full circuit around the axis of the cylinder (1.8) (see the explanation in the previous section) and, after reducing θ modulo T, returns to the neighbourhood of the singular point (x_0, θ_0) where the change of variables (2.2) acts.

At the final step, that is, after extending the function $v_1(\tau, \varepsilon, \delta)$ up to the value $\tau = \tau_{**}$, we uniquely determine the reserved parameter δ from the equation

$$v_1(\tau_{**}, \varepsilon, \delta) = v_2(\tau_{**}, \varepsilon, \delta). \tag{2.42}$$

We now begin the process of extension of $v_1(\tau, \varepsilon, \delta)$. Consider the interval $[\tau_*, \tau_1 - \varepsilon^{\lambda_2}]$, where $\lambda_2 = \mathrm{const} \in (0, 1/3)$. We stress that the properties (2.36) of the remainder R_1 certainly do not hold on this interval because $v_0(\tau) \to +\infty$ as $\tau \to \tau_1 - 0$. More precisely, by (2.27), (2.28) and the equalities $u_n(\tau_1) = u_n''(\tau_1) = 0$, we have an asymptotic expansion of the form

$$v_0(\tau) = \frac{1}{\tau_1 - \tau} + \sum_{k=1}^{\infty} s_k(\tau_1 - \tau)^k, \qquad \tau \to \tau_1 - 0. \tag{2.43}$$

For the function $v_{1,1}(\tau, \delta)$ we obtain from (2.32), (2.33), (2.43) that

$$v_{1,1}(\tau, \delta) = -\gamma_\eta'(0,0)\frac{\ln(\tau_1 - \tau)}{(\tau_1 - \tau)^2} + \frac{\omega_0(\delta)}{(\tau_1 - \tau)^2} + O\left(\frac{1}{\tau_1 - \tau}\right), \qquad \tau \to \tau_1 - 0, \tag{2.44}$$

where $\omega_0(\delta)$ is a smooth function of δ, whose explicit expression is irrelevant at this stage (its dependence on δ will be clarified below).

Thus we arrive at the following assertion.

Lemma 2.3. *On the interval $\tau_* \leqslant \tau \leqslant \tau_1 - \varepsilon^{\lambda_2}$ we have an asymptotic representation*

$$v_1(\tau, \varepsilon, \delta) = v_0(\tau) + \sqrt{\varepsilon}\, v_{1,1}(\tau, \delta) + \varepsilon^{1-\lambda_2} \ln^2 \frac{1}{\varepsilon} R_1(\tau, \varepsilon, \delta), \qquad \varepsilon \to 0, \tag{2.45}$$

where

$$|R_1(\tau, \varepsilon, \delta)| + \left| \frac{\partial R_1}{\partial \delta}(\tau, \varepsilon, \delta) \right| \leqslant \frac{M}{(\tau_1 - \tau)^2}. \tag{2.46}$$

Proof. We again consider the equation (2.37), where the function Φ has the old properties but h, Ψ are now such that

$$h(\tau,\varepsilon,\delta)=2v_0(\tau)+O\biggl(\sqrt{\varepsilon}\,\frac{\ln(\tau_1-\tau)}{(\tau_1-\tau)^2}\biggr),\qquad |\Psi|\leqslant M\,\frac{\ln^2(\tau_1-\tau)}{(\tau_1-\tau)^4}\tag{2.47}$$

uniformly in $\tau\in[\tau_*,\tau_1-\varepsilon^{\lambda_2}]$ and $\delta\in\Omega$. We pass from (2.37) to an integral equation (an analogue of (2.41)) written for the initial moment $\tau=\tau_*$ with initial condition

$$z(\tau_*,\varepsilon,\delta)=\varepsilon^{3/2-3\lambda_1}R_1(\tau_*,\varepsilon,\delta)=O(\varepsilon)$$

(see (2.36)). The properties (2.47) along with the condition $\lambda_2<1/3$ and the inequality

$$\exp\biggl[\int_\sigma^\tau h(\sigma',\varepsilon,\delta)\,d\sigma'\biggr]\leqslant M\biggl(\frac{\tau_1-\sigma}{\tau_1-\tau}\biggr)^2,\qquad \sigma\leqslant\tau,$$

which follows from the asymptotic formula for h (see (2.47)), guarantee that the contracting mapping principle is applicable to this integral equation on some subset K in $C([\tau_*,\tau_1-\varepsilon^{\lambda_2}])$ of the form

$$K=\biggl\{z(\tau)\colon |z(\tau)|\leqslant M\,\frac{\varepsilon^{1-\lambda_2}\ln^2(1/\varepsilon)}{(\tau_1-\tau)^2}\biggr\}.\qquad\square$$

At the next step we pass from $\tau=\tau_1-\varepsilon^{\lambda_2}$ to $\tau=\tau_1+\varepsilon^{\lambda_2}$. This is the most difficult step and, as already mentioned, the argument becomes non-local. The phase point $(x,\theta_1(x,\varepsilon,\mu))\big|_{\mu=\varepsilon\mu_n+\varepsilon^{3/2}\delta}$ first 'comes off' the singular point (x_0,θ_0) at a distance of order 1 (the rigorous meaning of this will be explained in Lemma 2.4). Then it moves quickly to some location at a close (but still of order 1) distance from (x_0,θ_0+T) and, finally, arrives at an asymptotically small neighbourhood of the point (x_0,θ_0+T) (this process is described in Lemma 2.5).

We now realize the programme outlined above. The extension of the function $v_1(\tau,\varepsilon,\delta)$ beyond the point $\tau=\tau_1-\varepsilon^{\lambda_2}$ is done by the change of variable

$$\xi=\sqrt{\varepsilon}\,\tau_1-\frac{1}{2}\gamma'_\eta(0,0)\varepsilon\ln\frac{1}{\varepsilon}+\varepsilon s.\tag{2.48}$$

Performing it in the normal form (2.6), we arrive at the equation

$$\frac{d\eta}{ds}=\bigl[\gamma(\xi,\eta)(\eta^2-\xi^2)+\varepsilon\Delta(\xi,\eta,\varepsilon,\sqrt{\varepsilon}\,\delta)\bigr]\big|_{\xi=\sqrt{\varepsilon}\,\tau_1-(1/2)\gamma'_\eta(0,0)\varepsilon\ln(1/\varepsilon)+\varepsilon s}.\tag{2.49}$$

By our constructions above, it must be endowed with the initial condition

$$\eta\big|_{s=s_0(\varepsilon)}=\overline{\eta}(\varepsilon,\delta),\tag{2.50}$$

where

$$s_0(\varepsilon)=-\varepsilon^{\lambda_2-1/2}+\frac{1}{2}\gamma'_\eta(0,0)\ln\frac{1}{\varepsilon},\qquad \overline{\eta}(\varepsilon,\delta)=\sqrt{\varepsilon}\,v_1(\tau_1-\varepsilon^{\lambda_2},\varepsilon,\delta).\tag{2.51}$$

We begin the asymptotic study of the solution $\eta(s,\varepsilon,\delta)$ of the Cauchy problem (2.49), (2.50) by finding the asymptotic behaviour of the initial condition $\overline{\eta}(\varepsilon,\delta)$

in (2.50) as $\varepsilon \to 0$. Therefore in what follows we always assume that the parameter λ_2 satisfies

$$\frac{1}{4} < \lambda_2 < \frac{1}{3}. \tag{2.52}$$

In view of (2.52), we substitute the expression (2.45) into the second equality (2.51) and use the properties (2.43), (2.44), (2.46). After some calculation we obtain that the following formulae hold uniformly in $\delta \in \Omega$ as $\varepsilon \to 0$:

$$\begin{gathered}\overline{\eta}(\varepsilon,\delta) = \varepsilon^{1/2-\lambda_2} + \biggl(\lambda_2\gamma'_\eta(0,0)\ln\frac{1}{\varepsilon} + \omega_0(\delta)\biggr)\varepsilon^{1-2\lambda_2} + O\biggl(\varepsilon^{3/2-3\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr),\\ \frac{\partial\overline{\eta}}{\partial\delta}(\varepsilon,\delta) = \omega'_0(\delta)\varepsilon^{1-2\lambda_2} + O\biggl(\varepsilon^{3/2-3\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr).\end{gathered} \tag{2.53}$$

We first consider the equation

$$\frac{d\eta}{ds} = \gamma(0,\eta)\eta^2, \tag{2.54}$$

which results from (2.49) with $\varepsilon = 0$, and its special solution $\eta_0(s) > 0$ that is determined for all negative s of sufficiently large modulus by the equation

$$Y(\eta) \stackrel{\text{def}}{=} -\frac{1}{\eta} - \gamma'_\eta(0,0)\ln|\eta| + \int_0^\eta \biggl[\frac{1}{\gamma(0,\sigma)\sigma^2} - \frac{1-\gamma'_\eta(0,0)\sigma}{\sigma^2}\biggr]\,d\sigma = s \tag{2.55}$$

and admits the following asymptotic expansion:

$$\eta_0(s) = -\frac{1}{s} - \gamma'_\eta(0,0)\,\frac{\ln(-s)}{s^2} + O\biggl(\frac{\ln^2(-s)}{s^3}\biggr), \qquad s \to -\infty. \tag{2.56}$$

Then the solution of the Cauchy problem (2.54), (2.50) is given by the formulae

$$\eta = \eta_0(s + \overline{s}(\varepsilon,\delta)), \qquad \overline{s}(\varepsilon,\delta) = Y(\overline{\eta}(\varepsilon,\delta)) - s_0(\varepsilon). \tag{2.57}$$

We also notice that in accordance with (2.51), (2.53) and the explicit expression for $Y(\eta)$ (see (2.55)), the following asymptotic formulae for the function $\overline{s}(\varepsilon,\delta)$ as $\varepsilon \to 0$ hold uniformly with respect to $\delta \in \Omega$:

$$\begin{gathered}\overline{s}(\varepsilon,\delta) = \omega_0(\delta) + O\biggl(\varepsilon^{1/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr),\\ \frac{\partial\,\overline{s}}{\partial\delta}(\varepsilon,\delta) = \omega'_0(\delta) + O\biggl(\varepsilon^{1/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr).\end{gathered} \tag{2.58}$$

Before stating rigorous results on the asymptotic behaviour of the solution $\eta(s,\varepsilon,\delta)$ of the problem (2.49), (2.50), we introduce another piece of notation. Recall that the function $\gamma(0,\eta)$ is defined only locally (on the closed interval $|\eta| \leqslant r$) and $\gamma(0,\eta) > 0$ (since $\gamma(0,0) = 1$). Fix an arbitrary value $r_* \in (0,r)$ and write $s_*(\delta)$ for the root of the equation $\eta_0(s + \omega_0(\delta)) = r_*$. The equalities (2.54), (2.56) and the properties of $\gamma(0,\eta)$ listed above show that this root is unique and the solution $\eta_0(s + \omega_0(\delta))$ is itself defined and monotone increasing from 0 to r_* on the semi-axis $-\infty < s \leqslant s_*(\delta)$.

Lemma 2.4. *Suppose that λ_2 satisfies the requirements (2.52). Then the following asymptotic representation as $\varepsilon \to 0$ holds on the closed interval $s_0(\varepsilon) \leqslant s \leqslant s_*(\delta)$:*

$$\eta(s,\varepsilon,\delta) = \eta_0(s+\omega_0(\delta)) + \varepsilon^{1/2-\lambda_2} \ln^2 \frac{1}{\varepsilon} R(s,\varepsilon,\delta), \tag{2.59}$$

where

$$|R(s,\varepsilon,\delta)| + \left|\frac{\partial R}{\partial \delta}(s,\varepsilon,\delta)\right| \leqslant \frac{M}{s^2+1}. \tag{2.60}$$

Proof. As usual, putting $\eta = \eta_0(s+\overline{s}(\varepsilon,\delta)) + z$ in (2.49), we arrive at the following equation for z:

$$\frac{dz}{ds} = h(s,\varepsilon,\delta)z + \Phi(z,s,\varepsilon,\delta) + \sqrt{\varepsilon}\,\Psi(s,\varepsilon,\delta). \tag{2.61}$$

By (2.56)–(2.58), the functions h, Φ, Ψ in (2.61) are such that

$$\begin{gathered} h(s,\varepsilon,\delta) = h_0(s,\delta) + O\biggl(\varepsilon^{1/2-\lambda_2} \ln^2 \frac{1}{\varepsilon}(s^2+1)^{-1}\biggr), \\ \Phi(0,s,\varepsilon,\delta) \equiv \frac{\partial \Phi}{\partial z}(0,s,\varepsilon,\delta) \equiv 0, \qquad \left|\frac{\partial \Phi}{\partial z}(z,s,\varepsilon,\delta)\right| \leqslant M|z| \quad \text{for} \quad |z| \leqslant 1, \\ |\Psi(s,\varepsilon,\delta)| \leqslant \frac{M}{s^2+1}, \end{gathered} \tag{2.62}$$

where

$$h_0(s,\delta) = \frac{d}{d\eta}\bigl(\gamma(0,\eta)\eta^2\bigr)\big|_{\eta=\eta_0(s+\omega_0(\delta))} = \frac{d^2\eta/ds^2}{d\eta/ds}\bigg|_{\eta=\eta_0(s+\omega_0(\delta))}. \tag{2.63}$$

We endow the equation (2.61) with the initial condition $z|_{s=s_0(\varepsilon)} = 0$ and pass from the resulting Cauchy problem to the integral equation

$$z(s,\varepsilon,\delta) = \int_{s_0(\varepsilon)}^{s} \exp\biggl[\int_\sigma^s h(\sigma',\varepsilon,\delta)\,d\sigma'\biggr]\bigl[\Phi(z,\sigma,\varepsilon,\delta) + \sqrt{\varepsilon}\,\Psi(\sigma,\varepsilon,\delta)\bigr]\,d\sigma. \tag{2.64}$$

Using (2.62), (2.63) and the bound

$$\exp\biggl[\int_\sigma^s h(\sigma',\varepsilon,\delta)\,d\sigma'\biggr] \leqslant M\biggl(\frac{d\eta_0(s+\omega_0(\delta))/ds}{d\eta_0(\sigma+\omega_0(\delta))/d\sigma}\biggr),$$

which follows from (2.63), we see that the right-hand side of (2.64) generates an operator that maps to itself some set $K \subset C([s_0(\varepsilon), s_*(\delta)])$ of the form

$$K = \biggl\{z(s)\colon |z(s)| \leqslant \frac{M\varepsilon^{\lambda_2}}{s^2+1}\biggr\} \tag{2.65}$$

and this map is a contraction (with constant of contraction of order $\varepsilon^{2\lambda_2-1/2}$). Thus the solution of the Cauchy problem (2.49), (2.50) satisfies

$$\eta(s,\varepsilon,\delta) = \eta_0(s+\overline{s}(\varepsilon,\delta)) + z(s,\varepsilon,\delta), \tag{2.66}$$

where $z(s,\varepsilon,\delta) \in K$ and $\partial z/\partial\delta \in K$. The relations (2.59), (2.60) are obtained from (2.66) by expanding the first term $\eta_0(s+\overline{s}(\varepsilon,\delta))$ with respect to ε. $\square$

We mention that the formulae (2.59), (2.60) hold on a larger interval of values of s. Namely, they hold for those $s \geqslant s_0(\varepsilon)$ which satisfy the *a priori* condition $|\eta(s,\varepsilon,\delta)| \leqslant r$. Taking this into account, we consider the equation

$$\eta(s,\varepsilon,\delta) = r_*. \tag{2.67}$$

Lemma 2.4 guarantees that it has a unique root $s = s_*(\varepsilon,\delta)$ with asymptotic behaviour (uniformly with respect to $\delta \in \Omega$)

$$\begin{gathered} s_*(\varepsilon,\delta) = s_*(\delta) + O\biggl(\varepsilon^{1/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr), \\ \frac{\partial s_*}{\partial \delta}(\varepsilon,\delta) = s_*'(\delta) + O\biggl(\varepsilon^{1/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr), \qquad \varepsilon \to 0. \end{gathered} \tag{2.68}$$

We further consider the point $(\xi,\eta) = (\xi_*, r_*)$, where

$$\xi_*(\varepsilon,\delta) = \sqrt{\varepsilon}\,\tau_1 - \frac{1}{2}\gamma_\eta'(0,0)\varepsilon\ln\frac{1}{\varepsilon} + \varepsilon s_*(\varepsilon,\delta), \tag{2.69}$$

and pass from it to the point (x_*,θ_*) by reversing the change of variables (2.2). Namely, $x_*(\varepsilon,\delta)$ is determined (as an implicit function) from the equation

$$\varphi_1(x) - \varphi_2(x) = 2\varkappa\,\xi_*(\varepsilon,\delta). \tag{2.70}$$

Once $x_*(\varepsilon,\delta)$ is found, the component $\theta_*(\varepsilon,\delta)$ is obtained from the formula

$$\theta_*(\varepsilon,\delta) = \frac{\varphi_1(x)+\varphi_2(x)}{2}\bigg|_{x=x_*(\varepsilon,\delta)} + \varkappa\, r_*. \tag{2.71}$$

Substituting the expansions (2.68), (2.69) into (2.70), (2.71) and making appropriate calculations, we obtain the following asymptotic representations (uniformly with respect to $\delta \in \Omega$) as $\varepsilon \to 0$:

$$\begin{gathered} x_*(\varepsilon,\delta) = x_0 + x_{1,*}\sqrt{\varepsilon} + x_{2,*}\varepsilon\ln\frac{1}{\varepsilon} + x_{3,*}(\delta)\varepsilon + O\biggl(\varepsilon^{3/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr), \\ \frac{\partial x_*}{\partial \delta}(\varepsilon,\delta) = x_{3,*}'(\delta)\varepsilon + O\biggl(\varepsilon^{3/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr); \end{gathered} \tag{2.72}$$

$$\theta_*(\varepsilon,\delta) = \theta_0 + \varkappa\, r_* + O(\sqrt{\varepsilon}\,), \qquad \frac{\partial \theta_*}{\partial \delta}(\varepsilon,\delta) = O(\varepsilon), \tag{2.73}$$

where

$$\begin{gathered} x_{1,*} = \frac{2\varkappa\,\tau_1}{\varphi_1'(x_0) - \varphi_2'(x_0)}, \qquad x_{2,*} = -\frac{\varkappa\,\gamma_\eta'(0,0)}{\varphi_1'(x_0) - \varphi_2'(x_0)}, \\ x_{3,*} = \frac{2\varkappa\, s_*(\delta)}{\varphi_1'(x_0) - \varphi_2'(x_0)} - \frac{2(\varphi_1''(x_0) - \varphi_2''(x_0))\varkappa^2\,\tau_1^2}{(\varphi_1'(x_0) - \varphi_2'(x_0))^3}. \end{gathered}$$

These constructions give rise to the Cauchy problem with initial condition $\theta|_{x=x_*} = \theta_*$ for the equation (1.1), (2.5). However, for technical reasons, it is more convenient to study the equivalent Cauchy problem

$$\frac{dx}{d\theta} = \frac{\varepsilon}{f(x,\theta,\varepsilon,\varepsilon\mu_n + \varepsilon^{3/2}\delta)}, \qquad x\big|_{\theta=\theta_*(\varepsilon,\delta)} = x_*(\varepsilon,\delta). \tag{2.74}$$

We consider the resulting problem on the closed interval $\theta_*(\varepsilon,\delta) \leqslant \theta \leqslant T$. It follows from Conditions 1.2, 1.3 that $f(x_0,\theta,0,0) > 0$ for $0 \leqslant \theta \leqslant T$, $\theta \neq \theta_0$. Combining this inequality with the expansions (2.72) and (2.73), we see that the Cauchy problem depends regularly on the parameters ε and δ. Hence the solution $x(\theta,\varepsilon,\delta)$ of this problem satisfies the following asymptotic equalities (uniformly with respect to $\delta \in \Omega$ and $\theta \in [\theta_*(\varepsilon,\delta),T]$) as $\varepsilon \to 0$:

$$\begin{gathered} x(\theta,\varepsilon,\delta) = x_*(\varepsilon,\delta) + \varepsilon \int_{\theta_0+\varkappa r_*}^{\theta} \frac{d\sigma}{f(x_0,\sigma,0,0)} + O(\varepsilon^{3/2}), \\ \frac{\partial x}{\partial\delta}(\theta,\varepsilon,\delta) = \frac{\partial x_*}{\partial\delta}(\varepsilon,\delta) + O(\varepsilon^{3/2}). \end{gathered} \tag{2.75}$$

For $\theta > T$ we make the change of variable $\theta - T \to \theta$ in (2.74) and use the periodicity of f with respect to θ. This yields the analogous Cauchy problem

$$\frac{dx}{d\theta} = \frac{\varepsilon}{f(x,\theta,\varepsilon,\varepsilon\mu_n + \varepsilon^{3/2}\delta)}, \qquad x\big|_{\theta=0} = x(T,\varepsilon,\delta), \tag{2.76}$$

whose solution will also be denoted by $x(\theta,\varepsilon,\delta)$. Since the problem (2.76) is regular, we see from the already-known expansions (2.75) that the following equalities hold (uniformly with respect to $\delta \in \Omega$ and $\theta \in [0,\overline{\theta}]$, where $\overline{\theta} = \mathrm{const} \in (0,\theta_0)$) as $\varepsilon \to 0$:

$$\begin{gathered} x(\theta,\varepsilon,\delta) = x_*(\varepsilon,\delta) + \varepsilon \int_{\theta_0+\varkappa r_*}^{T} \frac{d\sigma}{f(x_0,\sigma,0,0)} \\ + \varepsilon \int_0^{\theta} \frac{d\sigma}{f(x_0,\sigma,0,0)} + O(\varepsilon^{3/2}), \\ \frac{\partial x}{\partial\delta}(\theta,\varepsilon,\delta) = \frac{\partial x_*}{\partial\delta}(\varepsilon,\delta) + O(\varepsilon^{3/2}). \end{gathered} \tag{2.77}$$

At the next step we again pass from the variables (x,θ) to (ξ,η). Therefore, assuming that the parameter $\overline{\theta}$ belongs to the interval $(\theta_0 - \varkappa r_*, \theta_0)$, we consider the equation

$$\theta = \frac{\varphi_1(x) + \varphi_2(x)}{2}\bigg|_{x=x_*(\theta,\varepsilon,\delta)} - \varkappa r_* \tag{2.78}$$

and write $\theta = \theta_{**}(\varepsilon,\delta)$ for its root in the interval $0 \leqslant \theta \leqslant \overline{\theta}$. Then we put

$$x_{**}(\varepsilon,\delta) = x(\theta,\varepsilon,\delta)\big|_{\theta=\theta_{**}(\varepsilon,\delta)}, \qquad \xi_{**}(\varepsilon,\delta) = \frac{1}{2\varkappa}(\varphi_1(x) - \varphi_2(x))\big|_{x=x_{**}(\varepsilon,\delta)}. \tag{2.79}$$

In this case we easily see that the point $(\xi_{**}, -r_*)$, which corresponds to the point (x_{**},θ_{**}) in the coordinates (ξ,η), will be asymptotically close to $(0,-r_*)$.

We discuss separately the asymptotic calculation of $\xi_{**}(\varepsilon,\delta)$. Substituting the expansion (2.77) into (2.78), we see that, uniformly with respect to $\delta \in \Omega$,

$$\theta_{**}(\varepsilon,\delta) = \theta_0 - \varkappa r_* + O(\sqrt{\varepsilon}\,), \qquad \frac{\partial\theta_{**}}{\partial\delta}(\varepsilon,\delta) = O(\varepsilon)$$

as $\varepsilon \to 0$. Combining this with (2.77) and (2.79), we obtain the following asymptotic representations (uniformly with respect to $\delta \in \Omega$):

$$\begin{aligned}
\xi_{**}(\varepsilon,\delta) &= \sqrt{\varepsilon}\,\tau_1 - \frac{1}{2}\gamma'_\eta(0,0)\varepsilon \ln\frac{1}{\varepsilon} + \varepsilon s_{**}(\varepsilon,\delta),\\
s_{**}(\varepsilon,\delta) &= s_*(\delta) + \frac{\varphi'_1(x_0)-\varphi'_2(x_0)}{2\varkappa}\biggl(\int_{\theta_0+\varkappa r_*}^{T}\frac{d\sigma}{f(x_0,\sigma,0,0)}\\
&\qquad + \int_0^{\theta_0-\varkappa r_*}\frac{d\sigma}{f(x_0,\sigma,0,0)}\biggr) + O\biggl(\varepsilon^{1/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr),\\
\frac{\partial s_{**}}{\partial\delta}(\varepsilon,\delta) &= s'_*(\delta) + O\biggl(\varepsilon^{1/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr), \qquad \varepsilon\to 0.
\end{aligned} \tag{2.80}$$

When $\xi \geqslant \xi_{**}(\varepsilon,\delta)$ we must consider the Cauchy problem with initial condition $\eta|_{\xi=\xi_{**}} = -r_*$ for the equation (2.6). Performing the change of variables (2.48), we arrive at the Cauchy problem with initial condition

$$\eta\big|_{s=s_{**}(\varepsilon,\delta)} = -r_* \tag{2.81}$$

for the equation (2.49), where $s_{**}(\varepsilon,\delta)$ is the function in (2.80).

As above, to analyze the Cauchy problem (2.49), (2.81), we need a special solution $\eta=\eta_0(s)$ of (2.54). Consider the negative solution which is determined by the equation (2.55) for all sufficiently large $s>0$ and admits the following asymptotic representation (an analogue of (2.56)):

$$\eta_0(s) = -\frac{1}{s} - \gamma'_\eta(0,0)\frac{\ln s}{s^2} + O\biggl(\frac{\ln^2 s}{s^3}\biggr), \qquad s\to+\infty. \tag{2.82}$$

Using the function (2.82), we now construct a solution of the Cauchy problem (2.54), (2.81). It satisfies the following equalities (analogues of (2.57)):

$$\eta = \eta_0(s+\overline{\overline{s}}(\varepsilon,\delta)), \qquad \overline{\overline{s}}(\varepsilon,\delta) = Y(-r_*) - s_{**}(\varepsilon,\delta). \tag{2.83}$$

We easily see that the solution (2.83) is defined on the semi-axis $[s_{**}(\varepsilon,\delta),+\infty)$ and increases monotonically from $-r_*$ to 0 as s grows.

Before stating the next lemma, we consider the value

$$s_1(\varepsilon) = \varepsilon^{\lambda_2-1/2} + \frac{1}{2}\gamma'_\eta(0,0)\ln\frac{1}{\varepsilon} \tag{2.84}$$

of the variable s. It is analogous to the value $s=s_0(\varepsilon)$ in (2.51).

Lemma 2.5. *As above, assume that the parameter λ_2 satisfies the requirements (2.52). Then the solution $\eta(s,\varepsilon,\delta)$ of the Cauchy problem (2.49), (2.81) on the closed interval $s_{**}(\varepsilon,\delta)\leqslant s\leqslant s_1(\varepsilon)$ has the following asymptotic representation as $\varepsilon\to 0$:*

$$\eta(s,\varepsilon,\delta) = \eta_0(s+\psi_0(\delta)) + \varepsilon^{1/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\,R(s,\varepsilon,\delta), \tag{2.85}$$

where

$$\psi_0(\delta) = Y(-r_*) - s_*(\delta) \\ - \frac{\varphi_1'(x_0) - \varphi_2'(x_0)}{2\varkappa}\biggl(\int_{\theta_0+\varkappa r_*}^{T} \frac{d\sigma}{f(x_0,\sigma,0,0)} + \int_0^{\theta_0-\varkappa r_*} \frac{d\sigma}{f(x_0,\sigma,0,0)}\biggr), \tag{2.86}$$

$$|R(s,\varepsilon,\delta)| + \left|\frac{\partial R}{\partial \delta}(s,\varepsilon,\delta)\right| \leqslant \frac{M}{s^2+1}. \tag{2.87}$$

Proof. As in the proof of Lemma 2.4, we substitute the expression $\eta = \eta_0(s + \overline{\overline{s}}(\varepsilon,\delta)) + z$ into (2.49) and obtain an equation of the form (2.61) for z, where the functions h, Φ, Ψ still possess the properties (2.62), (2.63) (the solution $\eta = \eta_0(s+\omega_0(\delta))$ must be replaced in (2.63) by $\eta = \eta_0(s+\psi_0(\delta))$, where $\eta_0(s)$ and $\psi_0(\delta)$ are determined by the equations (2.82), (2.86)). We now pass from (2.61) to an integral equation analogous to (2.64), where the initial moment $s = s_0(\varepsilon)$ is replaced by $s = s_{**}(\varepsilon,\delta)$.

The analysis of the resulting equation repeats the corresponding fragment of the proof of Lemma 2.4 almost verbatim. Namely, we easily see that the right-hand side of the equation generates an operator which maps a certain set of functions $K \subset C([s_{**}(\varepsilon,\delta), s_1(\varepsilon)])$ of the form (2.65) to itself and this map is a contraction. Hence this operator has a unique fixed point $z(s,\varepsilon,\delta)$ such that

$$|z(s,\varepsilon,\delta)| + \left|\frac{\partial z}{\partial \delta}(s,\varepsilon,\delta)\right| \leqslant \frac{M\varepsilon^{\lambda_2}}{s^2+1}, \qquad s_{**}(\varepsilon,\delta) \leqslant s \leqslant s_1(\varepsilon). \tag{2.88}$$

The final formula (2.85) and the bound (2.87) are obtained from the equality

$$\eta(s,\varepsilon,\delta) = \eta_0(s + \overline{\overline{s}}(\varepsilon,\delta)) + z(s,\varepsilon,\delta)$$

using the bounds (2.88) and the representation

$$\eta_0(s + \overline{\overline{s}}(\varepsilon,\delta)) = \eta_0(s+\psi_0(\delta)) + O\biggl(\varepsilon^{1/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\,(s^2+1)^{-1}\biggr),$$

which follows from (2.80), (2.82), (2.83). □

At the next step we consider the equation (2.21) and endow it with the following initial condition (borrowed from the previous step):

$$v\big|_{\tau=\tau_1+\varepsilon^{\lambda_2}} = \overline{v}(\varepsilon,\delta) \overset{\text{def}}{=} \varepsilon^{-1/2}\eta(s,\varepsilon,\delta)\big|_{s=s_1(\varepsilon)}. \tag{2.89}$$

As above, we denote the solution of the resulting Cauchy problem by $v_1(\tau,\varepsilon,\delta)$.

We begin the study of this problem with an asymptotic analysis of the initial condition (2.89). Using (2.82) and (2.84)–(2.87), we obtain the following asymptotic equalities (uniformly with respect to $\delta \in \Omega$) as $\varepsilon \to 0$:

$$\overline{v}(\varepsilon,\delta) = -\varepsilon^{-\lambda_2} + \biggl(\lambda_2\gamma_\eta'(0,0)\ln\frac{1}{\varepsilon} + \psi_0(\delta)\biggr)\varepsilon^{1/2-2\lambda_2} + O\biggl(\varepsilon^{1-3\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr), \\ \frac{\partial\,\overline{v}}{\partial\delta}(\varepsilon,\delta) = \psi_0'(\delta)\varepsilon^{1/2-2\lambda_2} + O\biggl(\varepsilon^{1-3\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr). \tag{2.90}$$

We seek the asymptotic behaviour of $v_1(\tau,\varepsilon,\delta)$ in the following form, which is analogous to (2.30):

$$v_1 = v_0(\tau) + \sqrt{\varepsilon}\, v_{1,1}(\tau,\delta) + \cdots . \tag{2.91}$$

Here, as above, $v_0(\tau)$ is the function (2.28) and $v_{1,1}$ is a solution (analogous to (2.31)) of the linear inhomogeneous equation

$$\frac{dv_{1,1}}{d\tau} = 2v_0(\tau)v_{1,1} + g(\tau) \tag{2.92}$$

with inhomogeneous term (2.32).

We discuss separately the choice of a particular solution of (2.92). Notice from (2.28) that the general solution of the corresponding homogeneous equation is of the form

$$v_{1,1} = \frac{c}{u_n^2(\tau)}, \qquad c = \text{const} \in \mathbb{R}, \tag{2.93}$$

where u_n is the function in (2.27) and we have

$$v_{1,1} = \frac{c}{(u_n'(\tau_1))^2(\tau-\tau_1)^2} + O\biggl(\frac{1}{\tau-\tau_1}\biggr) \tag{2.94}$$

as $\tau \to \tau_1 + 0$. Furthermore, every solution $v_{1,1}(\tau)$ of the equation (2.92) satisfies the following asymptotic representation (analogous to (2.44)) in accordance with (2.93), (2.94) and the explicit formula (2.32) for the inhomogeneous term $g(\tau)$:

$$v_{1,1} = -\gamma_\eta'(0,0)\frac{\ln(\tau-\tau_1)}{(\tau-\tau_1)^2} + \frac{c}{(\tau-\tau_1)^2} + O\biggl(\frac{1}{\tau-\tau_1}\biggr), \qquad \tau \to \tau_1 + 0, \tag{2.95}$$

where the constant c may be arbitrary. Thus the choice of a concrete solution of (2.92) is completely determined by the choice of this constant.

In what follows we assume that the constant c in (2.95) is equal to $\psi_0(\delta)$ and denote the corresponding solution of (2.92) by $v_{1,1}(\tau,\delta)$. Using (2.93), (2.94), (2.32), we obtain the following equality for this solution:

$$\begin{aligned} v_{1,1}(\tau,\delta) = \frac{1}{u_n^2(\tau)}\biggl\{&(H)\int_{\tau_1}^{\tau} u_n^2(\sigma)v_0^3(\sigma)\,d\sigma\cdot\gamma_\eta'(0,0) \\ &+ \int_{\tau_1}^{\tau} u_n^2(\sigma)\Bigl[(v_0^2(\sigma)-\sigma^2)\sigma\gamma_\xi'(0,0) - \sigma^2 v_0(\sigma)\gamma_\eta'(0,0) + \sigma\Delta_\xi'(0,0,0,0) \\ &+ v_0(\sigma)\Delta_\eta'(0,0,0,0) + \delta\Delta_1(0,0,0,0)\Bigr]\,d\sigma\biggr\} + \frac{\psi_0(\delta)(u_n'(\tau_1))^2}{u_n^2(\tau)}, \end{aligned} \tag{2.96}$$

where $(H)\int_{\tau_1}^{\tau}$ is the Hadamard regularization (see [6]) at the point $\tau = \tau_1$.

The following assertion gives a rigorous meaning to the expansion (2.91).

Lemma 2.6. *As $\varepsilon \to 0$, we have an asymptotic representation of the form (2.45) whose remainder term $R_1(\tau,\varepsilon,\delta)$ satisfies (2.46) on the closed interval $\tau_1 + \varepsilon^{\lambda_2} \leqslant \tau \leqslant \overline{\tau}_1$, where $\overline{\tau}_1 = \text{const}$ in the interval (τ_1,τ_2).*

Proof. Putting $v = v_0 + \sqrt{\varepsilon}\, v_{1,1} + z$ in (2.21), where $v_{1,1}$ is now given by the formula (2.96), we obtain an equation of the form (2.37). Here the function Φ possesses the same properties as in (2.38) and h, Ψ satisfy (2.47) with $\tau_1 - \tau$ replaced by $\tau - \tau_1$. As usual, we pass from (2.37) to the corresponding integral equation, which now takes the form

$$\begin{aligned} z(\tau,\varepsilon,\delta) &= z(\tau_1+\varepsilon^{\lambda_2},\varepsilon,\delta)\exp\biggl\{\int_{\tau_1+\varepsilon^{\lambda_2}}^{\tau} h(\sigma,\varepsilon,\delta)\,d\sigma\biggr\} \\ &\quad + \int_{\tau_1+\varepsilon^{\lambda_2}}^{\tau} \exp\biggl\{\int_{\sigma}^{\tau} h(\sigma',\varepsilon,\delta)\,d\sigma'\biggr\}\bigl[\Phi(z,\sigma,\varepsilon,\delta)+\varepsilon\Psi(\sigma,\varepsilon,\delta)\bigr]\,d\sigma, \end{aligned} \tag{2.97}$$

where the following asymptotic formula as $\varepsilon \to 0$ holds uniformly with respect to $\delta \in \Omega$ because of (2.43), (2.90), (2.95), (2.96):

$$\begin{aligned} z(\tau_1+\varepsilon^{\lambda_2},\varepsilon,\delta) &\overset{\text{def}}{=} \overline{v}(\varepsilon,\delta) - [v_0(\tau)+\sqrt{\varepsilon}\, v_{1,1}(\tau,\delta)]\big|_{\tau=\tau_1+\varepsilon^{\lambda_2}} \\ &= O\biggl(\varepsilon^{1-3\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr), \\ \frac{\partial z}{\partial\delta}(\tau_1+\varepsilon^{\lambda_2},\varepsilon,\delta) &= O\biggl(\varepsilon^{1-3\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr). \end{aligned} \tag{2.98}$$

The analysis of the equation (2.97) uses the bounds

$$\exp\biggl[\int_{\tau_1+\varepsilon^{\lambda_2}}^{\tau} h(\sigma',\varepsilon,\delta)\,d\sigma'\biggr] \leqslant \frac{M\varepsilon^{2\lambda_2}}{(\tau-\tau_1)^2},$$

$$\exp\biggl[\int_{\sigma}^{\tau} h(\sigma',\varepsilon,\delta)\,d\sigma'\biggr] \leqslant M\biggl(\frac{\sigma-\tau_1}{\tau-\tau_1}\biggr)^2, \qquad \sigma\leqslant\tau,$$

which follow from the properties of the function h listed above, and the asymptotic representations (2.98). Combining these facts and applying the contraction mapping principle in the usual manner, we conclude that the equation has a unique solution

$$z = z(\tau,\varepsilon,\delta)\colon \quad |z(\tau,\varepsilon,\delta)| + \biggl|\frac{\partial z}{\partial\delta}(\tau,\varepsilon,\delta)\biggr| \leqslant M\,\frac{\varepsilon^{1-\lambda_2}\ln^2(1/\varepsilon)}{(\tau-\tau_1)^2}. \qquad \square$$

2.3. Conclusion of the justification of Theorem 1.1. Thus we have overcome the main difficulty in the proof of Theorem 1.1. Namely, the solution $v_1(\tau,\varepsilon,\delta)$ of (2.21) is extended beyond the singular point $\tau=\tau_1$, that is, to the interval $\tau_1+\varepsilon^{\lambda_2}\leqslant\tau\leqslant\overline{\tau}_1$. As a by-product, we obtained a series of basic Lemmas 2.1–2.6, which enable us to extend $v_1(\tau,\varepsilon,\delta)$ further by induction.

Indeed, suppose that the solution $v_1(\tau,\varepsilon,\delta)$ has already been extended to the interval $\tau_k+\varepsilon^{\lambda_2}\leqslant\tau\leqslant\overline{\tau}_k$ for some k, $1\leqslant k\leqslant n-1$, where $\overline{\tau}_k=\text{const}\in(\tau_k,\tau_{k+1})$ and the extension can be written in the form (2.45), (2.46) with τ_1 replaced by τ_k

and with the function $v_{1,1}(\tau,\delta)$ given the following equality (an analogue of (2.96)):

$$\begin{aligned} v_{1,1}(\tau,\delta) = \frac{1}{u_n^2(\tau)} \biggl\{ (H) \int_{\tau_k}^{\tau} u_n^2(\sigma) v_0^3(\sigma)\, d\sigma \cdot \gamma'_\eta(0,0) \\ + \int_{\tau_k}^{\tau} u_n^2(\sigma) \bigl[(v_0^2(\sigma) - \sigma^2)\sigma\gamma'_\xi(0,0) - \sigma^2 v_0(\sigma)\gamma'_\eta(0,0) + \sigma\Delta'_\xi(0,0,0,0) \\ + v_0(\sigma)\Delta'_\eta(0,0,0,0) + \delta\Delta_1(0,0,0,0) \bigr]\, d\sigma \biggr\} + \frac{\psi_{k-1}(\delta)(u'_n(\tau_k))^2}{u_n^2(\tau)}. \end{aligned} \tag{2.99}$$

Here $\psi_{k-1}(\delta)$ is a certain smooth function of δ, which is known from the previous constructions. We stress that all these facts have already been established for $k = 1$.

To make the inductive step, we first consider the values of τ in the interval $\overline{\tau}_k \leqslant \tau \leqslant \tau_{k+1} - \varepsilon^{\lambda_2}$. Notice that the function (2.99) satisfies the following asymptotic formula as $\tau \to \tau_{k+1} - 0$:

$$v_{1,1}(\tau,\delta) = -\gamma'_\eta(0,0) \frac{\ln(\tau_{k+1} - \tau)}{(\tau_{k+1} - \tau)^2} + \frac{\omega_k(\delta)}{(\tau_{k+1} - \tau)^2} + O\biggl(\frac{1}{\tau_{k+1} - \tau} \biggr), \tag{2.100}$$

where the quantity $\omega_k(\delta)$ is completely determined by the choice (2.99) of a particular solution of (2.92). Hence Lemma 2.3 can be applied. The only difference is that the initial condition for the function

$$z = v_1(\tau,\varepsilon,\delta) - v_0(\tau) - \sqrt{\varepsilon}\, v_{1,1}(\tau,\delta)$$

at $\tau = \overline{\tau}_k$ is now a quantity of order $\varepsilon^{1-\lambda_2}\ln^2(1/\varepsilon)$ instead of ε, as at $\tau = \tau_*$. But this difference does not affect the justification of the analogue of Lemma 2.3. Hence, for $\overline{\tau}_k \leqslant \tau \leqslant \tau_{k+1} - \varepsilon^{\lambda_2}$, we have

$$\begin{gathered} v_1(\tau,\varepsilon,\delta) = v_0(\tau) + \sqrt{\varepsilon}\, v_{1,1}(\tau,\delta) + \varepsilon^{1-\lambda_2} \ln^2 \frac{1}{\varepsilon} R_1(\tau,\varepsilon,\delta), \\ |R_1(\tau,\varepsilon,\delta)| + \biggl| \frac{\partial R_1}{\partial \delta}(\tau,\varepsilon,\delta) \biggr| \leqslant \frac{M}{(\tau_{k+1} - \tau)^2}. \end{gathered} \tag{2.101}$$

Further extension of $v_1(\tau,\varepsilon,\delta)$ is related to the change of variables (2.48). Performing it in the normal form (2.6), we arrive at the Cauchy problem (2.49), (2.50) with initial condition $\overline{\eta}(\varepsilon,\delta) = \sqrt{\varepsilon}\, v_1(\tau_{k+1} - \varepsilon^{\lambda_2},\varepsilon,\delta)$, which admits an asymptotic representation of the form (2.53) (with $\omega_0(\delta)$ replaced by $\omega_k(\delta)$) because of (2.100), (2.101). We write $s_*(\delta)$ for the root of the equation $\eta_0(s + \omega_k(\delta)) = r_*$, where $\eta_0(s)$ is the function in (2.56). Notice that this root is related to $\omega_k(\delta)$ by the equality

$$s_*(\delta) = \widetilde{s} - \omega_k(\delta), \tag{2.102}$$

where $\widetilde{s}$ is the root of the equation $\eta_0(s) = r_*$.

These facts guarantee that the hypotheses of Lemma 2.4 hold. Thus we obtain the following asymptotic representation (an analogue of (2.59), (2.60)) for the solution $\eta(s,\varepsilon,\delta)$ of the Cauchy problem (2.49), (2.50) on the interval $s_0(\varepsilon) \leqslant s \leqslant s_*(\delta)$, where $s_*(\delta)$ is the function (2.102):

$$\begin{gathered}\eta(s,\varepsilon,\delta)=\eta_0(s+\omega_k(\delta))+\varepsilon^{1/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\,R(s,\varepsilon,\delta),\\ |R(s,\varepsilon,\delta)|+\left|\frac{\partial R}{\partial\delta}(s,\varepsilon,\delta)\right|\leqslant\frac{M}{s^2+1}.\end{gathered}\tag{2.103}$$

Clearly, it follows from (2.103) that the equation (2.67) also admits a unique root $s=s_*(\varepsilon,\delta)$ with asymptotic behaviour (2.68). Thus we can repeat verbatim all the constructions described above, from the introduction of the point (ξ_*,r_*) (see (2.69)) to the construction of $(\xi_{**},-r_*)$ (see (2.80)). This in turn means that Lemma 2.5 still holds for the Cauchy problem (2.49), (2.81). By this lemma, the solution $\eta(s,\varepsilon,\delta)$ of this problem has the following asymptotic representation (an analogue of (2.85)–(2.87)):

$$\begin{gathered}\eta(s,\varepsilon,\delta)=\eta_0(s+\psi_k(\delta))+\varepsilon^{1/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\,R(s,\varepsilon,\delta),\\ |R(s,\varepsilon,\delta)|+\left|\frac{\partial R}{\partial\delta}(s,\varepsilon,\delta)\right|\leqslant\frac{M}{s^2+1},\qquad s_{**}(\varepsilon,\delta)\leqslant s\leqslant s_1(\varepsilon),\end{gathered}\tag{2.104}$$

where $\eta_0(s)$ is the function (2.82) and $\psi_k(\delta)$ is given by the following formula in view of (2.86) and (2.102):

$$\begin{aligned}\psi_k(\delta)&=Y(-r_*)+\omega_k(\delta)-\widetilde{s}\\ &\quad-\frac{\varphi_1'(x_0)-\varphi_2'(x_0)}{2\varkappa}\biggl(\int_{\theta_0+\varkappa r_*}^{T}\frac{d\sigma}{f(x_0,\sigma,0,0)}+\int_0^{\theta_0-\varkappa r_*}\frac{d\sigma}{f(x_0,\sigma,0,0)}\biggr).\end{aligned}\tag{2.105}$$

To complete the inductive step, we consider the interval $\tau_{k+1}+\varepsilon^{\lambda_2}\leqslant\tau\leqslant\overline{\tau}_{k+1}$, where $\overline{\tau}_{k+1}=\mathrm{const}\in(\tau_{k+1},\tau_{k+2})$ (when $k=n-1$, we assume that $\overline{\tau}_n=\tau_{**}$, $\overline{\tau}_{n+1}=+\infty$). On this interval we must consider the Cauchy problem for (2.21) with the following initial condition (analogous to (2.89)):

$$v\big|_{\tau=\tau_{k+1}+\varepsilon^{\lambda_2}}=\overline{v}(\varepsilon,\delta)\overset{\mathrm{def}}{=}\varepsilon^{-1/2}\eta(s,\varepsilon,\delta)\big|_{s=s_1(\varepsilon)}.\tag{2.106}$$

It follows from the relations (2.104), (2.105) that the function $\overline{v}(\varepsilon,\delta)$ in (2.106) satisfies the asymptotic formulae (2.90) (with $\psi_0(\delta)$ replaced by $\psi_k(\delta)$). Hence the problem (2.21), (2.106) can be analyzed using Lemma 2.6. By this lemma, the solution $v_1(\tau,\varepsilon,\delta)$ of this problem on the interval $\tau_{k+1}+\varepsilon^{\lambda_2}\leqslant\tau\leqslant\overline{\tau}_{k+1}$ admits a representation (2.45) with remainder term (2.46) (where τ_1 is replaced by τ_{k+1}) and the function $v_{1,1}(\tau,\delta)$ is given by an equality analogous to (2.99) (with k replaced by $k+1$).

The inductive process described above enables us to extend the solution $v_1(\tau,\varepsilon,\delta)$ to the interval $\tau_n+\varepsilon^{\lambda_2}\leqslant\tau\leqslant\tau_{**}$. We now make use of the reserved parameter δ, namely, we determine it from the equation (2.42). For convenience of our subsequent analysis, we write the required equation in the following form motivated by the known asymptotic properties of the functions $v_j(\tau,\varepsilon,\delta)$, $j=1,2$, near the value $\tau=\tau_{**}$ (see (2.36), (2.45), (2.46)):

$$v_{1,1}(\tau_{**},\delta)-v_{1,2}(\tau_{**},\delta)=\sqrt{\varepsilon}\,R_2(\tau_{**},\varepsilon,\delta)-\varepsilon^{1/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\,R_1(\tau_{**},\varepsilon,\delta),\tag{2.107}$$

where

$$|R_j(\tau_{**},\varepsilon,\delta)| + \left|\frac{\partial R_j}{\partial\delta}(\tau_{**},\varepsilon,\delta)\right| \leqslant M, \qquad j=1,2. \tag{2.108}$$

We first study the equation

$$v_{1,1}(\tau_{**},\delta) - v_{1,2}(\tau_{**},\delta) = 0, \tag{2.109}$$

which is obtained from (2.107) with $\varepsilon = 0$. More precisely, we shall prove that it is linear in δ. To do this, we rewrite it in the following form using the formulae (2.28), (2.35), (2.99) (with $k = n$):

$$\frac{\delta\Delta_1(0,0,0,0)}{u_n^2(\tau_{**})}\int_{\tau_n}^{+\infty} u_n^2(\sigma)\,d\sigma + \frac{\psi_{n-1}(\delta)(u_n'(\tau_n))^2}{u_n^2(\tau_{**})} + \text{const} = 0. \tag{2.110}$$

Here and in what follows we write const for quantities independent of δ.

By (2.110), we only need to study the dependence of $\psi_k(\delta)$, $k = 0,1,\dots,n-1$, on δ. Notice that since (2.32) is a linear function of δ, so is the coefficient $\omega_0(\delta)$ in (2.44). More precisely, it follows from (2.27), (2.28), (2.32), (2.33) that

$$\omega_0(\delta) = \frac{\delta\Delta_1(0,0,0,0)}{(u_n'(\tau_1))^2}\int_{-\infty}^{\tau_1} u_n^2(\sigma)\,d\sigma + \text{const}. \tag{2.111}$$

Combining this with (2.105), we conclude that

$$\psi_0(\delta) = \frac{\delta\Delta_1(0,0,0,0)}{(u_n'(\tau_1))^2}\int_{-\infty}^{\tau_1} u_n^2(\sigma)\,d\sigma + \text{const}. \tag{2.112}$$

Now suppose that $k \geqslant 1$. Then we see from (2.99), (2.105) that the coefficient $\omega_k(\delta)$ in the expansion (2.100) has the following structure:

$$\omega_k(\delta) = \frac{\delta\Delta_1(0,0,0,0)}{(u_n'(\tau_{k+1}))^2}\int_{\tau_k}^{\tau_{k+1}} u_n^2(\sigma)\,d\sigma + \frac{(u_n'(\tau_k))^2}{(u_n'(\tau_{k+1}))^2}\,\omega_{k-1}(\delta) + \text{const}.$$

Combining this with (2.105), (2.111), (2.112), we easily deduce that

$$\psi_{n-1}(\delta) = \frac{\delta\Delta_1(0,0,0,0)}{(u_n'(\tau_n))^2}\int_{-\infty}^{\tau_n} u_n^2(\sigma)\,d\sigma + \text{const}. \tag{2.113}$$

Substituting (2.113) into (2.110), we see that the equation (2.109) takes the form

$$\delta\Delta_1(0,0,0,0)\int_{-\infty}^{+\infty} u_n^2(\sigma)\,d\sigma + \text{const} = 0. \tag{2.114}$$

Since $\Delta_1(0,0,0,0) \neq 0$ (see (1.3), (2.4)), this equation has a unique solution $\delta = \delta_*$.

To complete the justification of Theorem 1.1, we shall assume that the compact set Ω of all possible values of δ is chosen in such a way that the value $\delta = \delta_*$ is an interior point of Ω. Then it follows in an obvious way from (2.108) that the equation (2.107) has a unique solution in Ω:

$$\delta = \delta_*(\varepsilon), \qquad \delta_*(\varepsilon) = \delta_* + O\biggl(\varepsilon^{1/2-\lambda_2}\ln^2\frac{1}{\varepsilon}\biggr), \quad \varepsilon \to 0.$$

Substituting this solution into (2.5), we obtain the required value $\mu = \mu_n(\varepsilon)$ such that the boundary-value problem (1.1), (1.9) is soluble. Theorem 1.1 is proved.

In conclusion we consider the situation when at least one of the conditions (1.10) does not hold. Since the equation (1.1) is T-periodic with respect to θ, there is no loss of generality in assuming that the quantities θ_1, θ_2 in the boundary conditions (1.9) belong to the intervals $0 \leqslant \theta < T$ and $0 < \theta \leqslant T$ respectively. Furthermore, we assume, for example, that

$$\varphi_2(a) < \theta_1 < T, \qquad \varphi_2(b) < \theta_2 \leqslant T. \tag{2.115}$$

Then for $n = 0$, the boundary-value problem (1.1), (1.9) has no solutions for all values of μ whose modulus is sufficiently small.

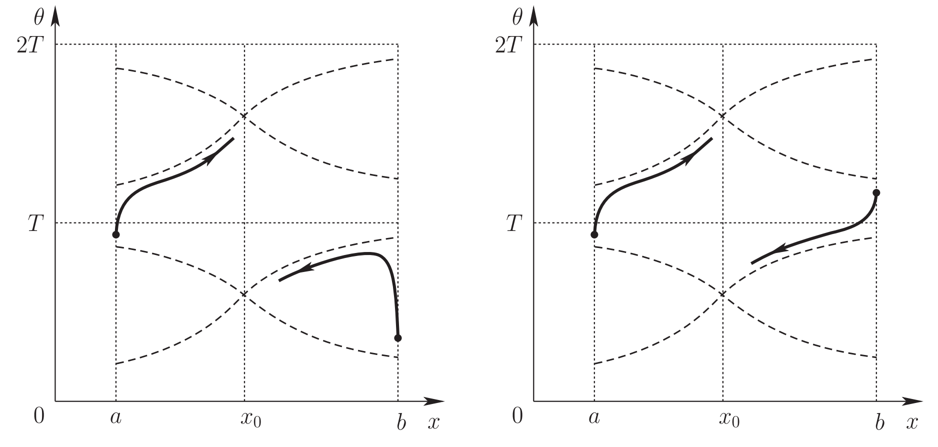

Figure 9 Figure 10

Indeed, in this case the solutions (2.1) of (1.1) behave as shown at Fig 9 when x increases or decreases respectively. The first solution asymptotically quickly reaches the half-plane $\theta > T$ (the phase point 'falls' onto the corresponding stable curve of slow motion) and stays there as x increases. The second solution lies in the half-plane $\theta < T$ for all x (it cannot intersect the line $\theta = T$ by Condition 1.2).

Our proof of Theorem 1.1 shows that the following analogue of this theorem holds for $n \geqslant 1$ and under the conditions (2.115). There is a unique value $\mu = \mu_n(\varepsilon)$, $\mu_n(0) = 0$, such that the boundary-value problem (1.1), (1.9) is soluble. However, the limit of the ratio $\mu_n(\varepsilon)/\varepsilon$ as $\varepsilon \to 0$ is now equal to μ_{n-1} instead of μ_n. The form of the limiting curve Γ_n also changes (the number of circuits decreases by 1). More precisely, in this case Γ_n consists of an $(n-1)$-circuit canard trajectory

$$\{(x,\theta)\colon a \leqslant x \leqslant x_0,\ \theta = \varphi_1(x) + T\} \cup \{(x,\theta)\colon x = x_0,\ \theta_0 + T \leqslant \theta \leqslant \theta_0 + nT\}$$
$$\cup\, \{(x,\theta)\colon x_0 \leqslant x \leqslant b,\ \theta = \varphi_1(x) + nT\},$$

a vertical interval $\{(x,\theta)\colon x = a,\ \theta_1 \leqslant \theta \leqslant \varphi_1(a) + T\}$ and another interval with endpoints $(b, \theta_2 + nT)$ and $(b, \varphi_1(b) + nT)$.

The same situation occurs in the case

$$0 \leqslant \theta_1 < \varphi_2(a), \qquad 0 < \theta_2 < \varphi_2(b), \tag{2.116}$$

when only the second condition in (1.10) does not hold. Namely, if $n = 0$ and the inequalities (2.116) hold, then the boundary-value problem (1.1), (1.9) has no solutions for all values of μ with sufficiently small modulus. But if $n \geqslant 1$, then an analogue of Theorem 1.1 holds with the following changes. The limiting value μ_n in (1.13) is replaced by μ_{n-1} and the curve Γ_n is modified in an appropriate way (as above, the number of its circuits is decreased by 1).

We now suppose that

$$\varphi_2(a) < \theta_1 < T, \qquad 0 < \theta_2 < \varphi_2(b). \tag{2.117}$$

Then the boundary-value problems (1.1), (1.9) with $n = 0$ and $n = 1$ have no solutions for all sufficiently small $|\mu|$. The case $n = 0$ has already been discussed and the behaviour of the solutions (2.1) in the case $n = 1$, as x increases or decreases, is shown in Fig. 10. The first solution asymptotically quickly reaches the half-plane $\theta > T$ (the phase point 'falls' onto the closest stable curve of slow motion) and never leaves it. The second solution asymptotically quickly reaches the half-plane $\theta < T$ and stays there as x decreases further.

Recalling the proof of Theorem 1.1 above, we conclude that an analogue of this theorem holds for $n \geqslant 2$ under the conditions (2.117). We must now replace the limiting value μ_n in (1.13) by μ_{n-2}. The curve Γ_n also changes its form. It now consists of an $(n-2)$-circuit canard trajectory connecting the curves of slow motion $\{(x,\theta)\colon \theta = \varphi_1(x) + T,\ a \leqslant x \leqslant x_0\}$, $\{(x,\theta)\colon \theta = \varphi_1(x) + (n-1)T,\ x_0 \leqslant x \leqslant b\}$, and of two vertical intervals

$$\{(x,\theta)\colon x = a,\ \theta_1 \leqslant \theta \leqslant \varphi_1(a) + T\},$$
$$\{(x,\theta)\colon x = b,\ \varphi_1(b) + (n-1)T \leqslant \theta \leqslant \theta_2 + nT\}.$$

§ 3. Other types of solutions

3.1. Description of results. In this section we study the behaviour of solutions of (1.1) for those values of μ that have order ε but are different from $\mu_n(\varepsilon)$, $n \geqslant 0$. Therefore we put

$$\mu = \mu_*(\alpha),$$
$$\mu_*(\alpha) \stackrel{\text{def}}{=} \frac{\alpha(\varphi_1'(x_0) - \varphi_2'(x_0)) + \varphi_1'(x_0) + \varphi_2'(x_0) - 2f_\varepsilon'(x_0,\theta_0,0,0)}{2f_\mu'(x_0,\theta_0,0,0)}, \tag{3.1}$$

in (1.1), where $\alpha = \text{const} \in \mathbb{R}$. Let $\theta(x,\varepsilon,\alpha)$ be a solution of the resulting equation with initial condition

$$\theta\Big|_{x=a} = \theta_1, \qquad 0 \leqslant \theta_1 < \varphi_2(a). \tag{3.2}$$

We first assume that $\alpha < 1$. Then it was shown in [14] that the curve

$$\Gamma(\varepsilon,\alpha) = \{(x,\theta)\colon a \leqslant x \leqslant b,\ \theta = \theta(x,\varepsilon,\alpha)\} \tag{3.3}$$

tends as $\varepsilon \to 0$ to the curve

$$\Gamma = \{(x,\theta)\colon a \leqslant x \leqslant x_0,\, \theta = \varphi_1(x)\} \cup \{(x,\theta)\colon x_0 \leqslant x \leqslant b,\, \theta = \varphi_2(x)\} \cup \Sigma, \quad (3.4)$$

where Σ is the closed vertical interval with endpoints (a,θ_1), $(a,\varphi_1(a))$. We stress that this curve, whose form is shown in Fig. 11, is not a canard trajectory since it contains no unstable parts of slow-motion curves.

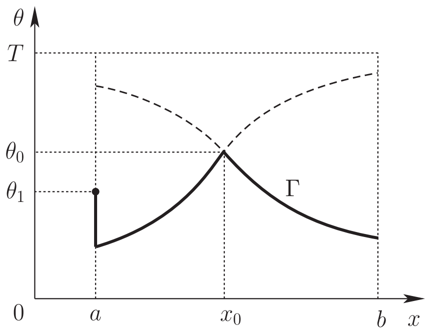

Figure 11

We now assume that

$$2n-1 < \alpha < 2n+1 \quad (3.5)$$

for some positive integer n. Then the following theorem holds.

Theorem 3.1. *Under the conditions* (3.5) *and for all sufficiently small* $\varepsilon > 0$, *the solution* $\theta(x,\varepsilon,\alpha)$ *of the Cauchy problem* (1.1), (3.1), (3.2) *is defined on the closed interval* $a \leqslant x \leqslant b$. *The limit of the curve* (3.3) *as* $\varepsilon \to 0$ *is the curve*

$$\begin{aligned}\Gamma = \{(x,\theta)\colon a \leqslant x \leqslant x_0,\, \theta = \varphi_1(x)\} \cup \{(x,\theta)\colon x = x_0,\, \theta_0 \leqslant \theta \leqslant \theta_0 + nT\}\\ \cup \{(x,\theta)\colon x_0 \leqslant x \leqslant b,\, \theta = \varphi_2(x) + nT\} \cup \Sigma, \quad (3.6)\end{aligned}$$

where Σ *is the interval in* (3.4).

The curve (3.6) is an n-circuit version of the curve (3.4). It is shown for $n = 1$ in Fig. 12. As in the case $n = 0$, it is not a canard trajectory.

In what follows we need an analogue of Theorem 3.1 for the Cauchy problem for the equation (1.1), (3.1) with initial condition

$$\theta\big|_{x=b} = \theta_2 + nT, \qquad \varphi_2(b) < \theta_2 \leqslant T. \quad (3.7)$$

Theorem 3.2. *Suppose that the inequalities* (3.5) *hold. Then, for all sufficiently small* $\varepsilon > 0$, *the solution* $\theta = \theta(x,\varepsilon,\alpha)$ *of the Cauchy problem* (1.1), (3.1), (3.7) *is defined on the closed interval* $a \leqslant x \leqslant b$ *and, as* $\varepsilon \to 0$, *the corresponding curve* (3.3) *tends to the curve*

$$\begin{aligned}\Gamma = \{(x,\theta)\colon x_0 \leqslant x \leqslant b,\, \theta = \varphi_1(x) + nT\}\\ \cup \{(x,\theta)\colon x = x_0,\, \theta_0 \leqslant \theta \leqslant \theta_0 + nT\}\\ \cup \{(x,\theta)\colon a \leqslant x \leqslant x_0,\, \theta = \varphi_2(x)\} \cup \Sigma. \quad (3.8)\end{aligned}$$

Here Σ *is the closed interval connecting the points* $(b,\theta_2 + nT)$ *and* $(b,\varphi_1(b) + nT)$.

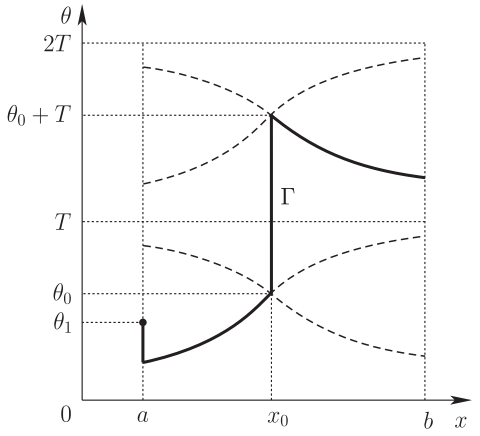

Figure 12

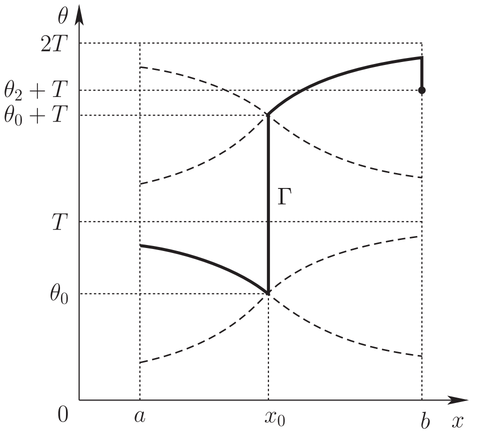

Figure 13

The limiting curve (3.8) for $n = 1$ is shown in Fig. 13. Unlike the two previous cases, it contains no stable fragments of slow-motion curves and, therefore, it is not a canard trajectory.

3.2. Proofs of Theorems 3.1 and 3.2. A principal role in the justification of Theorem 3.1 is played by the choice of a particular solution of the model equation (2.23). We have already mentioned the result in [14] that, for all negative τ of sufficiently large modulus, this equation has a unique solution $v = v_-(\tau, \alpha)$ with asymptotic behaviour (2.24). This is the solution to be studied in what follows.

It turns out that the function $v_-(\tau, \alpha)$ can be extended to the whole axis $\tau \in \mathbb{R}$ except for finitely many singularities of pole type. To see this, we fix the solution

$$u(\tau, \alpha) = |\tau|^{(\alpha-1)/2} \exp\biggl\{-\frac{\tau^2}{2} - \int_{-\infty}^{\tau} \biggl[v_-(\sigma, \alpha) - \sigma + \frac{\alpha - 1}{2\sigma}\biggr]\, d\sigma\biggr\} \tag{3.9}$$

of the linear equation (2.25). This solution is defined for all negative τ of sufficiently large modulus. By (2.24), we have

$$u(\tau, \alpha) = |\tau|^{(\alpha-1)/2} \exp\biggl(-\frac{\tau^2}{2}\biggr)\biggl[1 + O\biggl(\frac{1}{\tau^2}\biggr)\biggr], \qquad \tau \to -\infty. \tag{3.10}$$

We now extend the solution (3.9) to the whole axis $\tau \in \mathbb{R}$ and denote the extension again by $u(\tau, \alpha)$. Then the solution $v_-(\tau, \alpha)$ of (2.23) also extends to the whole axis (except for the zeros of $u(\tau, \alpha)$) by the formula

$$v_-(\tau, \alpha) = -\frac{1}{u(\tau, \alpha)} \frac{d}{d\tau} u(\tau, \alpha). \tag{3.11}$$

To establish additional properties of the function (3.11), we need some information about the solution $u(\tau, \alpha)$. Therefore we shall use the following assertion.

Lemma 3.1. *Suppose that the inequalities (3.5) hold. Then the solution $u(\tau, \alpha)$ has exactly n zeros $\tau_1(\alpha) < \tau_2(\alpha) < \cdots < \tau_n(\alpha)$ on the whole axis $\tau \in \mathbb{R}$ and* *satisfies the following asymptotic formula* (*which can be differentiated with respect to* τ) *as* $\tau \to +\infty$:

$$u(\tau,\alpha) = \frac{2^{-\alpha/2}\sqrt{2\pi}}{\Gamma(-\alpha/2+1/2)}\tau^{-(\alpha+1)/2}\exp\Big(\frac{\tau^2}{2}\Big)\Big[1+O\Big(\frac{1}{\tau^2}\Big)\Big], \tag{3.12}$$

where $\Gamma(z)$ *is the gamma function.*

Proof. By making the changes of variables

$$\tau = \frac{s}{\sqrt{2}}, \qquad \beta = -\frac{\alpha}{2} \tag{3.13}$$

in the equation (2.25), we bring it into the canonical form

$$\frac{d^2u}{ds^2} - \Big(\beta + \frac{s^2}{4}\Big)u = 0. \tag{3.14}$$

We shall use special solutions (see [20]) $U(\beta,s)$ and $V(\beta,s)$ of the resulting equation. The first solution satisfies the following asymptotic formula as $s \to +\infty$:

$$\begin{aligned} U(\beta,s) = s^{-\beta-1/2}\exp\Big(-\frac{s^2}{4}\Big)\Big[1 &- \frac{(\beta+1/2)(\beta+3/2)}{2s^2} \\ &+ \frac{(\beta+1/2)(\beta+3/2)(\beta+5/2)(\beta+7/2)}{2\cdot 4s^4} - \cdots\Big] \end{aligned} \tag{3.15}$$

and is uniquely determined by this requirement. The second solution is given by the equality

$$V(\beta,s) = \frac{1}{\pi}\Gamma\Big(\beta+\frac{1}{2}\Big)\big[U(\beta,s)\sin(\pi\beta) + U(\beta,-s)\big]. \tag{3.16}$$

It has the following asymptotic representation as $s \to +\infty$:

$$\begin{aligned} V(\beta,s) = \sqrt{\frac{2}{\pi}}\, s^{\beta-1/2}\exp\Big(\frac{s^2}{4}\Big)\Big[1 &+ \frac{(\beta-1/2)(\beta-3/2)}{2s^2} \\ &+ \frac{(\beta-1/2)(\beta-3/2)(\beta-5/2)(\beta-7/2)}{2\cdot 4s^4} + \cdots\Big]. \end{aligned} \tag{3.17}$$

To establish the asymptotic behaviour of $U(\beta,s)$ as $s \to -\infty$, we let s tend to $+\infty$ in (3.16) and use (3.15), (3.17). This yields that

$$\begin{aligned} U(\beta,s) = &-\sin(\pi\beta)|s|^{-\beta-1/2}\exp\Big(-\frac{s^2}{4}\Big)\Big[1+O\Big(\frac{1}{s^2}\Big)\Big] \\ &+ \frac{\sqrt{2\pi}}{\Gamma(\beta+1/2)}|s|^{\beta-1/2}\exp\Big(\frac{s^2}{4}\Big)\Big[1+O\Big(\frac{1}{s^2}\Big)\Big], \qquad s \to -\infty. \end{aligned} \tag{3.18}$$

For $V(\beta,s)$ we similarly deduce from the equality (see [20])

$$V(\beta,-s) = V(\beta,s)\sin(\pi\beta) + \frac{1}{\pi}\Gamma\Big(\beta+\frac{1}{2}\Big)\cos^2(\pi\beta)U(\beta,s)$$

that

$$V(\beta,s)=\sin(\pi\beta)\sqrt{\frac{2}{\pi}}\,|s|^{\beta-1/2}\exp\biggl(\frac{s^2}{4}\biggr)\biggl[1+O\biggl(\frac{1}{s^2}\biggr)\biggr]$$
$$+\frac{1}{\pi}\Gamma\biggl(\beta+\frac{1}{2}\biggr)\cos^2(\pi\beta)|s|^{-\beta-1/2}\exp\biggl(-\frac{s^2}{4}\biggr)\biggl[1+O\biggl(\frac{1}{s^2}\biggr)\biggr],\qquad s\to-\infty. \tag{3.19}$$

Recall that in view of (3.10), (3.13), we are interested in the solution $u=u(\beta,s)$ of (3.14) with prescribed asymptotic behaviour

$$u(\beta,s)=2^{\beta/2+1/4}|s|^{-\beta-1/2}\exp\biggl(-\frac{s^2}{4}\biggr)\biggl[1+O\biggl(\frac{1}{s^2}\biggr)\biggr],\qquad s\to-\infty. \tag{3.20}$$

Using (3.18) and (3.19), we easily see that this solution can be expressed in terms of $U(\beta,s)$ and $V(\beta,s)$ by the formulae

$$u(\beta,s)=2^{\beta/2+1/4}\biggl[\frac{\pi}{\Gamma(\beta+1/2)}V(\beta,s)-U(\beta,s)\sin(\pi\beta)\biggr]=2^{\beta/2+1/4}U(\beta,-s). \tag{3.21}$$

By the conditions (3.5) and the formulae (3.13), the parameter β lies in the interval

$$-\frac{1}{2}-n<\beta<\frac{1}{2}-n. \tag{3.22}$$

We shall now prove that, for these values of β, the function (3.21) has exactly n zeros on the whole axis $s\in\mathbb{R}$.

First, we note that, as β changes, new zeros of $u(\beta,s)$ can arise only from $+\infty$ (their bifurcation from $-\infty$ is impossible because (3.20) implies that $u(\beta,s)>0$ for all negative s of sufficiently large modulus). Furthermore, using the formula

$$u(\beta,s)\big|_{\beta=1/2-n}=2^{-(n-1)}(-1)^{n-1}\exp\biggl(-\frac{s^2}{4}\biggr)H_{n-1}\biggl(\frac{s}{\sqrt{2}}\biggr), \tag{3.23}$$

which follows from (3.21), we see that $u(\beta,s)$ has exactly $n-1$ zeros on the whole axis when $\beta=1/2-n$.

A bifurcation of the next zero of $u(\beta,s)$ from $+\infty$ occurs at

$$\beta=\frac{1}{2}-n-\nu,\qquad 0<\nu\ll 1. \tag{3.24}$$

Indeed, it follows from (3.15), (3.17) and (3.21) that

$$u(\beta,s)=2^{\beta/2+1/4}\biggl\{\frac{\sqrt{2\pi}}{\Gamma(\beta+1/2)}s^{\beta-1/2}\exp\biggl(\frac{s^2}{4}\biggr)\biggl[1+O\biggl(\frac{1}{s^2}\biggr)\biggr]$$
$$-\sin(\pi\beta)s^{-\beta-1/2}\exp\biggl(-\frac{s^2}{4}\biggr)\biggl[1+O\biggl(\frac{1}{s^2}\biggr)\biggr]\biggr\},\qquad s\to+\infty. \tag{3.25}$$

Then we substitute (3.24) into (3.25) and use the well-known property

$$\Gamma(-(n-1)-\nu)\sim\frac{(-1)^n}{\nu(n-1)!},\qquad \nu\to 0,$$

along with the fact that the asymptotic representations (3.15), (3.17) can be differentiated with respect to s. This yields that for all ν, $0 < \nu \ll 1$, the equation $u(1/2 - n - \nu, s) = 0$ has a unique root $s = s_n(\nu)$ on the semi-axis $s \geqslant \overline{s}$, where $\overline{s} > 0$ is a sufficiently large constant, and we have $s_n(\nu) \to +\infty$ as $\nu \to +0$. In the first approximation, this root coincides with the second positive root of the equation $s^{2n-1}\exp(-s^2/2) = \sqrt{2\pi}\,\nu(n-1)!\,$.

To summarize, it follows from the analysis above that the total number of roots of the equation (3.21) in the case (3.24) is equal to n. Indeed, by (3.23) this equation has $n-1$ zeros asymptotically close to the roots of the polynomial $H_{n-1}(s/\sqrt{2})$, and there is another root tending to $+\infty$ as $\nu \to +0$.

For every fixed value of β in the interval (3.22), no new zeros of $u(\beta, s)$ arise and no old zeros disappear. To see this, we consider the asymptotic representation

$$u(\beta, s) = 2^{\beta/2+1/4}\frac{\sqrt{2\pi}}{\Gamma(\beta+1/2)}s^{\beta-1/2}\exp\biggl(\frac{s^2}{4}\biggr)\biggl[1+O\biggl(\frac{1}{s^2}\biggr)\biggr], \qquad s \to +\infty, \tag{3.26}$$

which follows from (3.25), and take into account that $1/\Gamma(\beta+1/2) \neq 0$ in this case. Therefore the sign of $u(\beta, s)$ is preserved for all sufficiently large values of s. This in turn implies that no bifurcations (creating zeros from $+\infty$ or annihilating them at $+\infty$) can occur for these values of β.

We now return to the original equation (2.25) and note that its solutions $u(\tau, \alpha)$ are given by the equality

$$u(\tau, \alpha) = u(\beta, s)\big|_{\beta=-\alpha/2,\, s=\sqrt{2}\,\tau}. \tag{3.27}$$

It follows from the analysis above that the function (3.27) has exactly n zeros on the whole axis $\tau \in \mathbb{R}$. By (3.13) and (3.26), it also admits the required asymptotic representation (3.12) as $\tau \to +\infty$. □

We now proceed to prove Theorem 3.1. Note in this connection that the solution $\theta(x, \varepsilon, \alpha)$ of the problem (1.1), (3.1), (3.2) first behaves like the solution $\theta_1(x, \varepsilon, \mu)$ in (2.1). Namely, as x grows, the phase point $(x, \theta(x, \varepsilon, \alpha))$ 'falls' asymptotically quickly onto the stable curve $\theta = \varphi_1(x)$ of slow motion and then moves in the ε-neighbourhood of this curve until arriving at a sufficiently small neighbourhood of the singular point (x_0, θ_0).

As above, we write the equation (1.1) in the normal form (2.3) near the point (x_0, θ_0). Using the equation (3.1), we arrive at the following equation (an analogue of (2.6)):

$$\varepsilon\,\frac{d\eta}{d\xi} = \gamma(\xi, \eta)(\eta^2 - \xi^2) + \varepsilon\Delta(\xi, \eta, \varepsilon, \alpha) \tag{3.28}$$

with the remainder

$$\begin{gathered}\Delta(\xi, \eta, \varepsilon, \alpha) = \frac{1}{\varepsilon}\bigl(\mu\Delta_1(\xi, \eta, \varepsilon, \mu) + \varepsilon\Delta_2(\xi, \eta, \varepsilon, \mu)\bigr)\big|_{\mu=\mu_*(\alpha)\varepsilon},\\ \Delta(0, 0, 0, \alpha) \equiv \alpha.\end{gathered} \tag{3.29}$$

The function $\theta(x, \varepsilon, \alpha)$ induces a solution (an analogue of (2.9))

$$\eta = \mathcal{F}(\xi, \varepsilon, \alpha) \tag{3.30}$$

of the equation (3.28) on the interval $-q \leqslant \xi \leqslant -q_0$, where $q_0 > 0$ is sufficiently small. Moreover, by the results in [19], the following asymptotic formula holds uniformly with respect to $\xi \in [-q, -q_0]$ and $\alpha \in [2n-1, 2n+1]$:

$$\mathcal{F}(\xi, \varepsilon, \alpha) = \xi + \varepsilon a_1(\xi, \alpha) + O(\varepsilon^2), \qquad \varepsilon \to 0, \tag{3.31}$$

where, by (3.29),

$$\begin{gathered} a_1(\xi, \alpha) = \frac{1 - \Delta(\xi, \xi, 0, \alpha)}{2\xi\gamma(\xi, \xi)}, \qquad a_1(\xi, \alpha) = \frac{1-\alpha}{2\xi} + \varkappa_0 + O(\xi), \quad \xi \to -0, \\ \varkappa_0 = -\frac{1-\alpha}{2}\big(\gamma'_\xi(0,0) + \gamma'_\eta(0,0)\big) - \frac{1}{2}\big(\Delta'_\xi(0,0,0,\alpha) + \Delta'_\eta(0,0,0,\alpha)\big). \end{gathered} \tag{3.32}$$

The relations (3.31), (3.32) guarantee that an analogue of Lemma 2.1 is applicable to the solution (3.30). Therefore we fix $\lambda_1 \in (1/3, 1/2)$. By Lemma 2.1, the following asymptotic equality (an analogue of (2.14)) holds uniformly for $-q \leqslant \xi \leqslant -\varepsilon^{\lambda_1}$ and $2n-1 \leqslant \alpha \leqslant 2n+1$ as $\varepsilon \to 0$:

$$\mathcal{F}(\xi, \varepsilon, \alpha) = \xi + \varepsilon a_1(\xi, \alpha) + O(\varepsilon^{2-3\lambda_1}). \tag{3.33}$$

Our further analysis is related to the changes $\eta = \sqrt{\varepsilon}\, v$, $\xi = \sqrt{\varepsilon}\,\tau$, which transform the equation (3.28) into the following analogue of (2.21):

$$\frac{dv}{d\tau} = \gamma(\sqrt{\varepsilon}\,\tau, \sqrt{\varepsilon}\, v)(v^2 - \tau^2) + \Delta(\sqrt{\varepsilon}\,\tau, \sqrt{\varepsilon}\, v, \varepsilon, \alpha). \tag{3.34}$$

We write $v(\tau, \varepsilon, \alpha)$ for the solution of this equation with initial condition

$$v\big|_{\tau = -\varepsilon^{\lambda_1 - 1/2}} = \varepsilon^{-1/2}\mathcal{F}(-\varepsilon^{\lambda_1}, \varepsilon, \alpha) \tag{3.35}$$

and seek its asymptotic representation in the form analogous to (2.30):

$$v = v_-(\tau, \alpha) + \sqrt{\varepsilon}\, v_1(\tau, \alpha) + \cdots, \tag{3.36}$$

where $v_-(\tau, \alpha)$ is the solution of (2.23) chosen above. The function $v_1(\tau, \alpha)$ in (3.36) is defined by the following equality (an analogue of (2.33)):

$$v_1(\tau, \alpha) = \int_{-\infty}^{\tau} \exp\biggl[\int_\sigma^\tau 2v_-(\sigma', \alpha)\, d\sigma'\biggr] g(\sigma, \alpha)\, d\sigma, \tag{3.37}$$

where

$$\begin{aligned} g(\tau, \alpha) = \big(v_-^2(\tau, \alpha) - \tau^2\big)\big(\tau\gamma'_\xi(0,0) + v_-(\tau, \alpha)\gamma'_\eta(0,0)\big) + \tau\Delta'_\xi(0,0,0,\alpha) \\ + v_-(\tau, \alpha)\Delta'_\eta(0,0,0,\alpha). \end{aligned} \tag{3.38}$$

Using (3.37), (3.38) and the asymptotic representation (2.24), we easily see that

$$v_1(\tau, \alpha) = \varkappa_0 + O\biggl(\frac{1}{\tau}\biggr), \qquad \tau \to -\infty,$$

where $\varkappa_0$ is the constant in (3.32). This in turn implies that

$$\varepsilon^{-1/2}\mathcal{F}(-\varepsilon^{\lambda_1}, \varepsilon, \alpha) - \big(v_-(\tau, \alpha) + \sqrt{\varepsilon}\, v_1(\tau, \alpha)\big)\big|_{\tau = -\varepsilon^{\lambda_1 - 1/2}} = O(\varepsilon^{3/2 - 3\lambda_1}). \tag{3.39}$$

The resulting equality (3.39) guarantees that an analogue of Lemma 2.2 is applicable to the solution of the Cauchy problem (3.34), (3.35). To use this lemma, we fix an arbitrary $\tau_* \in (-\infty, \tau_1(\alpha))$, where $\tau_1(\alpha)$ is the first zero of $u(\tau,\alpha)$. Thus we see that the following asymptotic representation (as $\varepsilon \to 0$) of the solution $v = v(\tau,\varepsilon,\alpha)$ of the problem mentioned above holds uniformly with respect to $\tau \in [-\varepsilon^{\lambda_1-1/2}, \tau_*]$ and $\alpha \in \Omega$, where $\Omega \subset (2n-1, 2n+1)$ is an arbitrary compact set:

$$\begin{gathered} v(\tau,\varepsilon,\alpha) = v_-(\tau,\alpha) + \sqrt{\varepsilon}\, v_1(\tau,\alpha) + \varepsilon^{3/2-3\lambda_1} R(\tau,\varepsilon,\alpha), \\ |R| \leqslant M\left(\frac{u(-\varepsilon^{\lambda_1-1/2},\alpha)}{u(\tau,\alpha)}\right)^2 + \varepsilon^{3\lambda_1-1/2} M(|\tau|+1) \end{gathered} \tag{3.40}$$

(as above, we use the same letter M for various universal positive constants whose exact values are irrelevant).

It is important to note that, by Lemma 3.1 and (3.11), the function $v_-(\tau,\alpha)$ satisfies the following asymptotic equalities (analogues of (2.43)) at the points $\tau_1(\alpha) < \tau_2(\alpha) < \cdots < \tau_n(\alpha)$:

$$v_-(\tau,\alpha) = \frac{1}{\tau_k(\alpha)-\tau} + O(\tau_k(\alpha)-\tau), \qquad \tau \to \tau_k(\alpha), \quad k = 1,\dots,n. \tag{3.41}$$

The properties (3.41) enable us to use Lemmas 2.3–2.6 and extend the function $v(\tau,\varepsilon,\alpha)$ beyond the point $\tau = \tau_n(\alpha)$. As in the proof of Theorem 1.1, we successively define this function on the closed intervals $[\tau_*, \tau_1(\alpha)-\varepsilon^{\lambda_2}]$, $[\tau_k(\alpha)+\varepsilon^{\lambda_2}, \tau_{k+1}(\alpha)-\varepsilon^{\lambda_2}]$, $k = 1,\dots,n-1$, $[\tau_n(\alpha)+\varepsilon^{\lambda_2}, \tau_{**}]$, where τ_{**} is a constant in the interval $(\tau_n(\alpha), +\infty)$ and the constant λ_2 satisfies (2.52). When we pass from the value $\tau = \tau_k(\alpha)-\varepsilon^{\lambda_2}$ to $\tau = \tau_k(\alpha)+\varepsilon^{\lambda_2}$, the phase point $(x, \theta(x,\varepsilon,\alpha))$ makes one full circuit around the axis of the cylinder (1.8).

We omit the detailed analysis, which repeats the corresponding fragment of the proof of Theorem 1.1 almost verbatim, and state the final result. In this case, the following asymptotic equality (as $\varepsilon \to 0$) holds uniformly with respect to $\tau \in [\tau_n(\alpha)+\varepsilon^{\lambda_2}, \tau_{**}]$ and $\alpha \in \Omega$:

$$\begin{gathered} v(\tau,\varepsilon,\alpha) = v_-(\tau,\alpha) + \sqrt{\varepsilon}\, v_1(\tau,\alpha) + \varepsilon^{1-\lambda_2}\ln^2\frac{1}{\varepsilon}\, R(\tau,\varepsilon,\alpha), \\ |R| \leqslant \frac{M}{(\tau-\tau_n(\alpha))^2}. \end{gathered} \tag{3.42}$$

Here, as in (3.40), $v_-(\tau,\alpha)$ is the function (3.11) and $v_1(\tau,\alpha)$ is a certain particular solution of the equation

$$\frac{dv}{d\tau} = 2v_-(\tau,\alpha)v + g(\tau,\alpha), \tag{3.43}$$

where $g(\tau,\alpha)$ is the function (3.38). This solution can be written in the form

$$v_1(\tau,\alpha) = \frac{1}{u^2(\tau,\alpha)}\int_{\tau_{**}}^{\tau} u^2(\sigma,\alpha)g(\sigma,\alpha)\,d\sigma + \frac{\widetilde{v}(\alpha)}{u^2(\tau,\alpha)}, \tag{3.44}$$

where the coefficient $\widetilde{v}(\alpha)$ is known from the previous constructions and its explicit expression is irrelevant (only its continuity with respect to α is important).

When $\tau_{**} \leqslant \tau \leqslant \varepsilon^{\lambda_1-1/2}$, where $\lambda_1 \in (1/3, 1/2)$, we encounter the first new aspects compared to the proof of Theorem 1.1. The solutions $v_\pm(\tau,\alpha)$ of (2.23) are not conjugate in this case. Hence no equation of the form (2.24) holds for $v_-(\tau,\alpha)$ as $\tau \to +\infty$. By (3.11) and (3.12), we obtain instead that

$$v_-(\tau,\alpha) = -\tau + \frac{\alpha+1}{2\tau} + O\biggl(\frac{1}{\tau^3}\biggr), \qquad \tau \to +\infty. \tag{3.45}$$

Combining (3.12) and (3.45) and using them in (3.38), (3.44), we conclude that the solution (3.44) of the equation (3.43) satisfies the following asymptotic equality:

$$v_1(\tau,\alpha) = \varkappa_0 + O\biggl(\frac{1}{\tau}\biggr), \qquad \tau \to +\infty, \tag{3.46}$$

where

$$\varkappa_0 = -\frac{1+\alpha}{2}\bigl(\gamma'_\xi(0,0) - \gamma'_\eta(0,0)\bigr) + \frac{1}{2}\bigl(\Delta'_\xi(0,0,0,\alpha) - \Delta'_\eta(0,0,0,\alpha)\bigr). \tag{3.47}$$

The results (3.45)–(3.47) enable us to establish the following lemma.

Lemma 3.2. *The solution* (3.42) *on the interval* $\tau_{**} \leqslant \tau \leqslant \varepsilon^{\lambda_1-1/2}$ *admits the following asymptotic representation as* $\varepsilon \to 0$:

$$v(\tau,\varepsilon,\alpha) = v_-(\tau,\alpha) + \sqrt{\varepsilon}\, v_1(\tau,\alpha) + \varepsilon^{1-\lambda_2}\ln^2\frac{1}{\varepsilon}\, R(\tau,\varepsilon,\alpha), \tag{3.48}$$

where the remainder R *can be estimated as*

$$|R(\tau,\varepsilon,\alpha)| \leqslant \frac{M}{u^2(\tau,\alpha)} + \frac{\varepsilon^{\lambda_2}}{\ln^2(1/\varepsilon)} M(|\tau|+1). \tag{3.49}$$

Proof. Substituting the equality

$$v = v_-(\tau,\alpha) + \sqrt{\varepsilon}\, v_1(\tau,\alpha) + z$$

into (3.34), we arrive at the following equation (analogous to (2.37)) for finding z:

$$\frac{dz}{d\tau} = h(\tau,\varepsilon,\alpha)z + \Phi(z,\tau,\varepsilon,\alpha) + \varepsilon\Psi(\tau,\varepsilon,\alpha), \tag{3.50}$$

where, as in (2.38), the functions h, Φ, Ψ possess the properties

$$\begin{gathered} h(\tau,\varepsilon,\alpha) = 2v_-(\tau,\alpha) + O\bigl(\sqrt{\varepsilon}(\tau^2+1)\bigr), \\ \Phi(0,\tau,\varepsilon,\alpha) \equiv \frac{\partial\Phi}{\partial z}(0,\tau,\varepsilon,\alpha) \equiv 0, \\ \biggl|\frac{\partial\Phi}{\partial z}(z,\tau,\varepsilon,\alpha)\biggr| \leqslant M|z| \quad \text{for} \quad |z| \leqslant 1, \\ |\Psi(\tau,\varepsilon,\alpha)| \leqslant M(|\tau|+1)^2 \end{gathered} \tag{3.51}$$

uniformly with respect to $\tau \in [\tau_{**}, \varepsilon^{\lambda_1-1/2}]$ and $\alpha \in \Omega$. As usual, we endow the equation (3.50) with initial conditions

$$z\big|_{\tau=\tau_{**}} = \varepsilon^{1-\lambda_2}\ln^2\frac{1}{\varepsilon}\, R(\tau_{**},\varepsilon,\alpha),$$

where R is the function in (3.42). As a result, we arrive at the integral equation

$$z(\tau,\varepsilon,\alpha) = \varepsilon^{1-\lambda_2}\ln^2\frac{1}{\varepsilon}\,R(\tau_{**},\varepsilon,\alpha)\exp\biggl[\int_{\tau_{**}}^{\tau} h(\sigma,\varepsilon,\alpha)\,d\sigma\biggr] + \int_{\tau_{**}}^{\tau}\exp\biggl[\int_{\sigma}^{\tau} h(\sigma',\varepsilon,\delta)\,d\sigma'\biggr]\bigl[\Phi(z,\sigma,\varepsilon,\alpha)+\varepsilon\Psi(\sigma,\varepsilon,\alpha)\bigr]\,d\sigma. \tag{3.52}$$

The analysis of the equation (3.52) is based on the bounds

$$\int_{\tau_{**}}^{\tau}\exp\biggl[\int_{\sigma}^{\tau} h(\sigma',\varepsilon,\delta)\,d\sigma'\biggr](|\sigma|+1)^k\,d\sigma \leqslant M(|\tau|+1)^{k-1}, \qquad k=0,1,2,$$

$$|R(\tau_{**},\varepsilon,\alpha)|\leqslant M, \quad \exp\biggl[\int_{\sigma}^{\tau} h(\sigma',\varepsilon,\delta)\,d\sigma'\biggr]\leqslant M\biggl(\frac{u(\sigma,\alpha)}{u(\tau,\alpha)}\biggr)^2, \qquad \sigma\leqslant\tau,$$

which follow from (3.42), (3.45)–(3.47), (3.51). The facts listed enable us to conclude that the operator generated by the right-hand side of (3.52) maps to itself a certain subset K of $C([\tau_{**},\varepsilon^{\lambda_1-1/2}])$ of the form

$$K=\biggl\{z(\tau)\colon |z(\tau)|\leqslant \frac{M}{u^2(\tau,\alpha)}\,\varepsilon^{1-\lambda_2}\ln^2\frac{1}{\varepsilon}+\varepsilon M(|\tau|+1)\biggr\}$$

and this map is a contraction (with contraction constant of order $\varepsilon^{1-\lambda_2}\ln^2(1/\varepsilon)$). Hence the equation mentioned has a unique solution $z(\tau,\varepsilon,\alpha)\in K$. □

We further extend the solution $\theta(x,\varepsilon,\alpha)$ with respect to x using the equation (3.28) along with the initial condition

$$\eta\big|_{\xi=\varepsilon^{\lambda_1}}=\sqrt{\varepsilon}\,v(\tau,\varepsilon,\alpha)\big|_{\tau=\varepsilon^{\lambda_1-1/2}}. \tag{3.53}$$

Let $\eta=\mathcal{F}(\xi,\varepsilon,\alpha)$ be the solution of the resulting Cauchy problem. We first notice that the function $\eta=-\xi+\varepsilon a_1(\xi,\alpha)$, where

$$a_1(\xi,\alpha)=\frac{1+\Delta(\xi,-\xi,0,\alpha)}{2\xi\gamma(\xi,-\xi)}, \tag{3.54}$$

satisfies (3.28) up to a discrepancy of order $O(\varepsilon^2/\xi^2)$. Secondly, by (3.45)–(3.49) and (3.54) we have

$$\sqrt{\varepsilon}\,v(\tau,\varepsilon,\alpha)\big|_{\tau=\varepsilon^{\lambda_1-1/2}}-\bigl(-\xi+\varepsilon a_1(\xi,\alpha)\bigr)\big|_{\xi=\varepsilon^{\lambda_1}}=O(\varepsilon^{2-3\lambda_1}).$$

Using this and arguing as in the proof of Lemma 2.1, we easily deduce that the following asymptotic representation (as $\varepsilon\to 0$) holds for the solution of the Cauchy problem (3.28), (3.53) uniformly with respect to (ξ,α) such that $\varepsilon^{\lambda_1}\leqslant\xi\leqslant q$ and $\alpha\in\Omega$:

$$\mathcal{F}(\xi,\varepsilon,\alpha)=-\xi+\varepsilon a_1(\xi,\alpha)+O(\varepsilon^{2-3\lambda_1}). \tag{3.55}$$

The equality (3.55) means that at $\xi=q$ we arrive at an asymptotically small neighbourhood of the slow-motion stable curve $\eta=-\xi$. When we pass from (ξ,η) to the original coordinates (x,θ), the point $(q,-q)$ is mapped to the point

$(x_*, \varphi_2(x_*)+nT)$, where $x_* > x_0$ is close to x_0, and the curve $\eta = -\xi$ is mapped to the curve $\theta = \varphi_2(x)+nT$ (we recall that n full circuits around the axis of the cylinder (1.8) were performed as x passed through the singular point x_0). By the results in [19], further motion of the phase point $(x, \theta(x, \varepsilon, \alpha))$ occurs in the asymptotically small neighbourhood of the stable 'slow' curve $\{(x,\theta)\colon x_* \leqslant x \leqslant b,\ \theta = \varphi_2(x)+nT\}$ up to the final point $x = b$.

As a result of our asymptotic analysis, we conclude that the curve (3.3) tends to the curve (3.6) as $\varepsilon \to 0$. Theorem 3.1 is proved.

We notice that Theorem 3.2 requires no separate proof since it reduces to Theorem 3.1 by the changes $x \to -x$, $\theta \to -\theta$. Then all the unstable manifolds of slow motion become stable and vice versa. The function (3.1) remains the same because the quantities $\varphi_1'(x_0)$, $\varphi_2'(x_0)$ are the roots of the polynomial in (1.5), which is invariant under these changes.

To conclude, we note that analogues of Theorems 3.1, 3.2 hold in the cases when the inequalities $\varphi_2(a) < \theta_1 < T$ and $0 < \theta_2 < \varphi_2(b)$ hold in (3.2) and (3.7) respectively. But then the number of circuits of the corresponding curves (3.6), (3.8) decreases by one and the vertical closed intervals Σ in these curves are replaced by $\Sigma = \{(x,\theta)\colon x = a,\ \theta_1 \leqslant \theta \leqslant \varphi_1(a)+T\}$ in the first case and $\Sigma = \{(x,\theta)\colon x = b,$ $\varphi_1(b) + (n-1)T \leqslant \theta \leqslant \theta_2 + nT\}$ in the second.

§ 4. Singularly perturbed spectral problems

4.1. The Schrödinger equation. It turns out that Theorems 1.1, 3.1, 3.2, which were established above, can be applied in the asymptotic analysis of the spectrum of boundary-value problems of Sturm–Liouville type for a singularly perturbed linear Schrödinger equation. The corresponding result will be stated below.

Consider the following singularly perturbed spectral problem on a bounded interval $a \leqslant x \leqslant b$:

$$\begin{gathered} -\varepsilon^2 \frac{d^2u}{dx^2} + q(x,\varepsilon)u = \mu u, \\ u\Big|_{x=a} \cos\theta_1 - \varepsilon \frac{du}{dx}\Big|_{x=a} \sin\theta_1 = 0, \qquad u\Big|_{x=b} \cos\theta_2 - \varepsilon \frac{du}{dx}\Big|_{x=b} \sin\theta_2 = 0. \end{gathered} \tag{4.1}$$

Here $\varepsilon > 0$ is a small parameter, $q(x,\varepsilon) \in C^\infty([a,b] \times [0,\varepsilon_0])$, where $\varepsilon_0 > 0$ is sufficiently small, μ is the spectral parameter and the quantities θ_1, θ_2 satisfy the conditions $0 \leqslant \theta_1 < \pi$, $0 < \theta_2 \leqslant \pi$.

The main restriction in our study of the problem (4.1) will be imposed on the potential $q_0(x) = q(x,0)$.

Condition 4.1. *There are points $x_1 < x_2 < \cdots < x_n$ in the interval (a,b) such that*

$$\begin{gathered} q_0(x_k) = q_0'(x_k) = 0, \quad q_0''(x_k) > 0, \qquad k = 1, \dots, n; \\ q_0(x) > 0 \quad \text{for} \quad x \neq x_k, \qquad k = 1, \dots, n. \end{gathered} \tag{4.2}$$

The relations (4.2) mean that the potential $q(x,\varepsilon)$ for $\varepsilon = 0$ has n potential wells at level zero and is positive at the other points of the interval $a \leqslant x \leqslant b$.

To state the next restriction, we define the numbers

$$\mu_{k,i} = q_1(x_k) + (2i+1)\sqrt{\frac{1}{2}q_0''(x_k)}, \qquad k = 1,\dots,n, \quad i = 0,1,\dots, \tag{4.3}$$

where $q_1(x) = \partial q(x,\varepsilon)/\partial\varepsilon\big|_{\varepsilon=0}$.

Condition 4.2. *The numbers* (4.3) *are pairwise distinct.*

It is known [21] that the eigenvalues of the boundary-value problem (4.1) form a monotone increasing sequence

$$\mu_0(\varepsilon) < \mu_1(\varepsilon) < \dots < \mu_j(\varepsilon) < \cdots, \tag{4.4}$$

which tends to $+\infty$ as $j \to \infty$. Moreover, every eigenfunction $u_j(x,\varepsilon)$ corresponding to the jth eigenvalue, has exactly j zeros on the interval (a,b). Let μ_j, $j \geqslant 0$, be the sequence of numbers (4.3) arranged in increasing order.

Theorem 4.1. *Suppose that Conditions 4.1, 4.2 hold and the constants θ_1, θ_2 in* (4.1) *are such that*

$$0 \leqslant \theta_1 < \frac{\pi}{2} + \arctan\sqrt{q_0(a)}, \qquad \frac{\pi}{2} - \arctan\sqrt{q_0(b)} < \theta_2 \leqslant \pi. \tag{4.5}$$

Then for every fixed $j \geqslant 0$ we have the following limit equality for the jth eigenvalue (4.4):

$$\lim_{\varepsilon\to 0}\frac{\mu_j(\varepsilon)}{\varepsilon} = \mu_j. \tag{4.6}$$

Proof. We pass to the polar coordinates (ρ,θ) in (4.1) by making the following standard change of variables (see [21]):

$$\varepsilon\frac{du}{dx} = \rho\cos\theta, \qquad u = \rho\sin\theta. \tag{4.7}$$

Then the equation for θ takes a form analogous to (1.1):

$$\varepsilon\frac{d\theta}{dx} = \cos^2\theta - q(x,\varepsilon)\sin^2\theta + \mu\sin^2\theta. \tag{4.8}$$

We now discuss some properties of the equation (4.8) that follow from Condition 4.1 and will be used below. First, when $\varepsilon = \mu = 0$ it has two slow-motion curves $\theta = f_k(x)$, $k = 1, 2$, of class $C^\infty([a,b])$ in the rectangle $\Pi_0 = \{(x,\theta)\colon a \leqslant x \leqslant b,\ 0 < \theta < \pi\}$. They are given by the formulae

$$\begin{gathered} f_1(x) = \frac{\pi}{2} - (-1)^j\arctan\sqrt{q_0(x)}, \qquad f_2(x) = \frac{\pi}{2} - (-1)^{j+1}\arctan\sqrt{q_0(x)} \\ \text{for}\quad x_j \leqslant x \leqslant x_{j+1}, \quad j = 0,1,\dots,n, \qquad \text{where}\quad x_0 = a,\ x_{n+1} = b. \end{gathered} \tag{4.9}$$

The mutual position of these curves when $n = 2$ is shown in Fig. 14 (the stable parts are shown by the solid lines and the unstable parts by the broken line). Second, the curves (4.9) intersect each other in a generic way at the points $(x_k, \pi/2)$, $k = 1,\dots,n$, and analogues of Conditions 1.1–1.3 hold in a neighbourhood of each

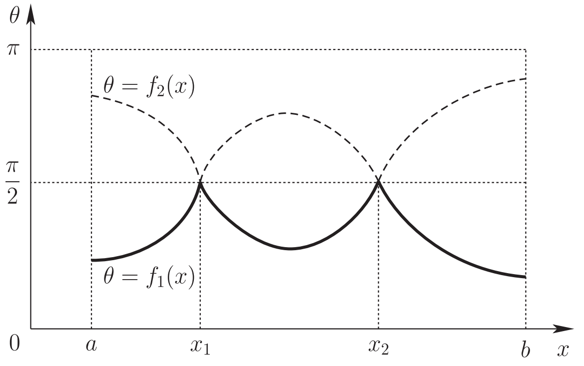

Figure 14

intersection point. Third, the numbers (4.3) coincide with the quantities (1.11) evaluated at $n = i$ for every singular point $x = x_k$.

We fix an arbitrary integer $j \geqslant 0$ and consider the jth eigenvalue (4.4). By the results in [21] and (4.7), the boundary-value problem for the equation (4.8) with boundary conditions

$$\theta\big|_{x=a} = \theta_1, \qquad \theta\big|_{x=b} = \theta_2 + \pi j, \tag{4.10}$$

where θ_1, θ_2 are borrowed from (4.1), is soluble for $\mu = \mu_j(\varepsilon)$. Hence the task of justifying the limit relation (4.6) reduces to an asymptotic analysis of this boundary-value problem.

Here is a plan of what follows. Consider integers k_0, i_0 such that

$$\mu_j = \mu_{k_0,i_0} \tag{4.11}$$

(the uniqueness of this pair of numbers is guaranteed by Condition 4.2). Then put

$$\mu = \varepsilon\mu_{k_0,i_0} + \varepsilon^{3/2}\delta \tag{4.12}$$

in (4.8), where the parameter δ varies over an arbitrary fixed compact set Ω. Let $\theta = \theta_r(x, \varepsilon, \delta)$, $r = 1, 2$, be the solutions of the resulting equation with initial conditions

$$\theta_1(x, \varepsilon, \delta)\big|_{x=a} = \theta_1, \qquad \theta_2(x, \varepsilon, \delta)\big|_{x=b} = \theta_2 + \pi j.$$

The parameter δ in (4.12) will be chosen to 'match' these solutions near the point $x = x_{k_0}$.

We first consider the solution $\theta_1(x, \varepsilon, \delta)$. It follows from the first condition in (4.5) and the location of the slow-motion curves (4.9) (see Fig. 14) that, as x grows, the phase point $(x, \theta_1(x, \varepsilon, \delta))$ 'falls' asymptotically quickly onto the stable curve of slow motion

$$\theta = \frac{\pi}{2} - \arctan\sqrt{q_0(x)}, \qquad a \leqslant x < x_1, \tag{4.13}$$

and then moves in a neighbourhood of this curve to a point asymptotically close to $(x_1, \pi/2)$.

For every k, $1 \leqslant k \leqslant n$, $k \neq k_0$, let j_k be the cardinality of the set of those numbers (4.3) that are smaller than (4.11). Then $\mu_{k,i} < \mu_{k_0,i_0}$ for $i = 0, 1, \dots, j_k - 1$ and $\mu_{k,j_k} > \mu_{k_0,i_0}$. In particular, when $k = 1$ we have

$$\mu_{1,j_1-1} < \mu_{k_0,i_0} < \mu_{1,j_1}. \tag{4.14}$$

Notice that the inequalities (4.14) are equivalent to relations of the form (3.1), (3.5) (if we replace x_0 by x_1 and n by j_1). Hence, under the condition (4.12), Theorem 3.1 is applicable in a neighbourhood of the point $(x_1, \pi/2)$.

By this theorem, the solution $\theta_1(x, \varepsilon, \delta)$ 'comes off' the manifold (4.13) near $x = x_1$. Then it moves asymptotically quickly in an asymptotically small neighbourhood of the vertical interval $\{(x, \theta)\colon x = x_1,\ \pi/2 \leqslant \theta \leqslant \pi/2 + \pi j_1\}$ and arrives at the next stable curve of slow motion

$$\theta = \frac{\pi}{2} - \arctan \sqrt{q_0(x)} + \pi j_1, \qquad x_1 < x < x_2.$$

We argue further by induction. Suppose that for some k, $k \leqslant k_0 - 1$, moving in an asymptotically small neighbourhood of the curve

$$\theta = \frac{\pi}{2} - \arctan \sqrt{q_0(x)} + \pi \sum_{s=1}^{k-1} j_s, \qquad x_{k-1} < x < x_k, \tag{4.15}$$

the phase point $(x, \theta_1(x, \varepsilon, \delta))$ arrives at an asymptotically small neighbourhood of the next singular point

$$\biggl(x_k, \frac{\pi}{2} + \pi \sum_{s=1}^{k-1} j_s\biggr).$$

Then the following analogues of (4.14) hold by the definition of j_k:

$$\mu_{k,j_k-1} < \mu_{k_0,i_0} < \mu_{k,j_k}.$$

Hence Theorem 3.1 is again applicable. By this theorem, the phase point 'comes off' the curve (4.15) at $x = x_k$ and then moves asymptotically quickly along the interval

$$\biggl\{(x, \theta)\colon x = x_k,\ \frac{\pi}{2} + \pi \sum_{s=1}^{k-1} j_s \leqslant \theta \leqslant \frac{\pi}{2} + \pi \sum_{s=1}^{k} j_s\biggr\}$$

and arrives at the stable curve of slow motion

$$\theta = \frac{\pi}{2} - \arctan \sqrt{q_0(x)} + \pi \sum_{s=1}^{k} j_s, \qquad x_k < x < x_{k+1}.$$

It follows from this recursive process that the curve $(x, \theta_1(x, \varepsilon, \delta))$ can be extended to any sufficiently small fixed neighbourhood of the point

$$\biggl(x_{k_0}, \frac{\pi}{2} + \pi \sum_{s=1}^{k_0-1} j_s\biggr).$$

A symmetric situation occurs in case of the solution $\theta_2(x,\varepsilon,\delta)$, but we now use Theorem 3.2 instead of Theorem 3.1. Indeed, by the second condition in (4.5), as x decreases, the phase point $(x,\theta_2(x,\varepsilon,\delta))$ 'falls' onto the curve

$$\theta=\frac{\pi}{2}+\arctan\sqrt{q_0(x)}+\pi j, \qquad x_n<x\leqslant b,$$

and, moving in an asymptotically small neighbourhood of this curve, arrives at a neighbourhood of the singular point $(x_n,\pi/2+\pi j)$. Then, by Theorem 3.2, it moves quickly along the interval

$$\Bigl\{(x,\theta)\colon x=x_n,\ \frac{\pi}{2}+\pi j-\pi j_n\leqslant\theta\leqslant\frac{\pi}{2}+\pi j\Bigr\}$$

and continues a slow motion in an asymptotically small neighbourhood of the curve

$$\theta=\frac{\pi}{2}+\arctan\sqrt{q_0(x)}+\pi j-\pi j_n, \qquad x_{n-1}<x<x_n.$$

As in the previous case, we establish by induction that the curve $(x,\theta_2(x,\varepsilon,\delta))$ can finally be extended to any fixed sufficiently small neighbourhood of the point

$$\biggl(x_{k_0},\ \frac{\pi}{2}+\pi j-\pi\sum_{s=0}^{n-k_0-1}j_{n-s}\biggr).$$

These constructions reduce the problem of the solubility of the boundary-value problem (4.8), (4.10) to analyzing the boundary-value problem for the equation (4.8), (4.12) with boundary conditions

$$\theta\big|_{x=x_{k_0}-\sigma_0}=\theta_1(x_{k_0}-\sigma_0,\varepsilon,\delta), \qquad \theta\big|_{x=x_{k_0}+\sigma_0}=\theta_2(x_{k_0}+\sigma_0,\varepsilon,\delta). \tag{4.16}$$

Here $\sigma_0>0$ is a fixed sufficiently small constant and, by the constructions above and the results in [19], the functions $\theta_1(x_{k_0}-\sigma_0,\varepsilon,\delta)$ and $\theta_2(x_{k_0}+\sigma_0,\varepsilon,\delta)$ satisfy the following asymptotic representations (as $\varepsilon\to0$) uniformly with respect to $\delta\in\Omega$:

$$\begin{gathered}\theta_1(x_{k_0}-\sigma_0,\varepsilon,\delta)=\frac{\pi}{2}-\arctan\sqrt{q_0(x_{k_0}-\sigma_0)}+\pi\sum_{s=1}^{k_0-1}j_s+O(\varepsilon),\\ \theta_2(x_{k_0}+\sigma_0,\varepsilon,\delta)=\frac{\pi}{2}+\arctan\sqrt{q_0(x_{k_0}+\sigma_0)}+\pi j-\pi\sum_{s=0}^{n-k_0-1}j_{n-s}+O(\varepsilon).\end{gathered} \tag{4.17}$$

These formulae can be differentiated with respect to δ. Furthermore, since the equation (4.8) is π-periodic with respect to θ and we have

$$\sum_{\substack{s=1\\ s\neq k_0}}^{n}j_s=j-i_0,$$

the boundary conditions (4.16) admit a factorization modulo π. As a result, they take the form

$$\theta\big|_{x=x_{k_0}-\sigma_0}=\theta_1(\varepsilon,\delta), \qquad \theta\big|_{x=x_{k_0}+\sigma_0}=\theta_2(\varepsilon,\delta)+\pi i_0, \tag{4.18}$$

where, by (4.17),

$$\begin{aligned}\theta_1(\varepsilon,\delta) &\stackrel{\text{def}}{=} \theta_1(x_{k_0}-\sigma_0,\varepsilon,\delta)-\pi\sum_{s=1}^{k_0-1} j_s \\ &= \frac{\pi}{2}-\arctan\sqrt{q_0(x_{k_0}-\sigma_0)}+O(\varepsilon), \\ \theta_2(\varepsilon,\delta) &\stackrel{\text{def}}{=} \theta_2(x_{k_0}+\sigma_0,\varepsilon,\delta)-\pi j+\pi\sum_{s=0}^{n-k_0-1} j_{n-s} \\ &= \frac{\pi}{2}+\arctan\sqrt{q_0(x_{k_0}+\sigma_0)}+O(\varepsilon).\end{aligned} \tag{4.19}$$

At the final step, we apply Theorem 1.1 to the boundary-value problem (4.8), (4.12), (4.18). We stress that all the hypotheses of this theorem hold. In particular, analogues of the inequalities (1.10) hold by (4.19). Thus there is a unique value $\delta=\delta(\varepsilon)\sim 1$, $\varepsilon\to 0$, such that the boundary-value problem is soluble. □

We now clarify the geometric meaning of our constructions. Let $\theta_j(x,\varepsilon)$ be a solution of the boundary-value problem (4.8), (4.10). It follows from the justification of Theorem 4.1 given above that the curve $\{(x,\theta)\colon a\leqslant x\leqslant b,\ \theta=\theta_j(x,\varepsilon)\}$ contains a canard trajectory as a fragment. More precisely, the fragment corresponding to the values $x\in[x_{k_0}-\sigma_0,x_{k_0}+\sigma_0]$ for a sufficiently small $\sigma_0>0$ is a canard trajectory that connects the slow-motion curves

$$\theta=\frac{\pi}{2}-\arctan\sqrt{q_0(x)}+\pi\sum_{s=1}^{k_0-1} j_s, \qquad \theta=\frac{\pi}{2}+\arctan\sqrt{q_0(x)}+\pi j-\pi\sum_{s=0}^{n-k_0-1} j_{n-s}$$

and makes i_0 circuits, where k_0, i_0 are the numbers in (4.11). The neighbourhoods of the other singular points $x=x_k$, $k=1,\dots,n$, $k\neq k_0$, contain no canard trajectories. The graph of a typical function $\theta=\theta_j(x,\varepsilon)$ is shown in Fig. 15. It corresponds to the case when $j=2$, $n=3$, $k_0=3$, $i_0=1$, $\mu_{1,0}<\mu_{3,1}<\mu_{1,1}$, $\mu_{3,1}<\mu_{2,0}$.

We discuss separately what happens when at least one of the conditions (4.5) does not hold. For example, suppose that

$$\frac{\pi}{2}+\arctan\sqrt{q_0(a)}<\theta_1<\pi, \qquad \frac{\pi}{2}-\arctan\sqrt{q_0(b)}<\theta_2\leqslant\pi. \tag{4.20}$$

Then we can apply analogues of Theorems 1.1, 3.1 to the variant (2.115). These results and the constructions performed in the proof of Theorem 4.1 yield the following assertion.

Theorem 4.2. *If Conditions 4.1, 4.2 and the inequalities (4.20) hold, then we have the following limit relations (analogues of (4.6)):*

$$\lim_{\varepsilon\to 0}\frac{\mu_j(\varepsilon)}{\varepsilon}=\mu_{j-1}, \qquad j\geqslant 1. \tag{4.21}$$

The question of the asymptotic behaviour of the eigenvalue $\mu_0(\varepsilon)$ is not covered by (4.21) and is a separate problem related to the analysis of the boundary-value problem (4.8), (4.10) with $j=0$. We assume that the parameter μ in this problem

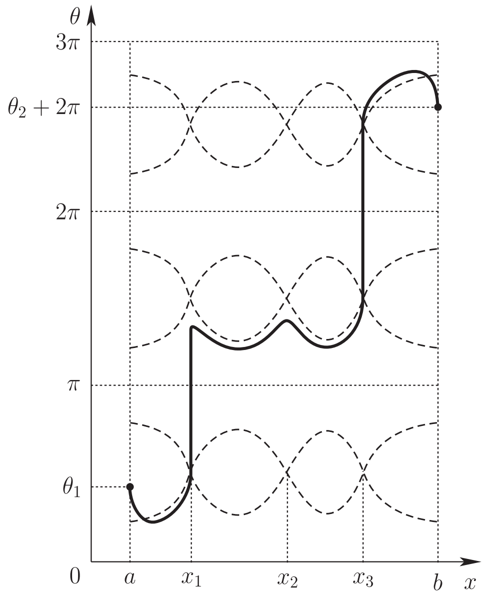

Figure 15

is a negative quantity of order 1. Then, when $\varepsilon = 0$, the equation (4.8) has an unstable manifold of slow motion

$$\theta = \frac{\pi}{2} + \arctan\sqrt{q_0(x) - \mu}, \qquad a \leqslant x \leqslant b, \tag{4.22}$$

which passes through the point $(x, \theta) = (a, \theta_1)$ at

$$\mu = \overline{\mu} \stackrel{\text{def}}{=} q_0(a) - \cot^2\theta_1 < 0. \tag{4.23}$$

Let $\theta = \theta(x, \varepsilon, \mu)$ be a solution of (4.8) with initial condition $\theta\big|_{x=b} = \theta_2$. By the second condition in (4.20), as x decreases, this solution 'falls' onto the slow-motion curve (4.22) and stays asymptotically close to it for $a \leqslant x < b$ (see Fig. 16). Clearly, an appropriate 'perturbation' of the parameter μ in a neighbourhood of the value (4.23) enables us to assume that $\theta(a, \varepsilon, \mu) = \theta_1$. This means that

$$\lim_{\varepsilon \to 0} \mu_0(\varepsilon) = \overline{\mu}. \tag{4.24}$$

The remaining two cases

$$0 \leqslant \theta_1 < \frac{\pi}{2} + \arctan\sqrt{q_0(a)}, \qquad 0 < \theta_2 < \frac{\pi}{2} - \arctan\sqrt{q_0(b)} \tag{4.25}$$

and

$$\frac{\pi}{2} + \arctan\sqrt{q_0(a)} < \theta_1 < \pi, \qquad 0 < \theta_2 < \frac{\pi}{2} - \arctan\sqrt{q_0(b)} \tag{4.26}$$

are similar to the case discussed above. Therefore we only state the final results. Under the conditions (4.25), the relation (4.21) still holds and (4.24) is replaced

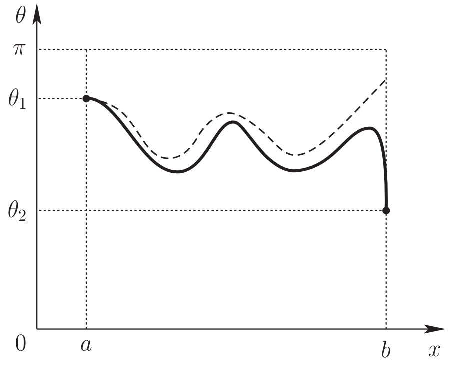

Figure 16

by the equation

$$\lim_{\varepsilon\to 0}\mu_0(\varepsilon)=\overline{\overline{\mu}}, \qquad \overline{\overline{\mu}} \overset{\text{def}}{=} q_0(b)-\cot^2\theta_2<0. \tag{4.27}$$

In the case (4.26) we have

$$\begin{gathered}\lim_{\varepsilon\to 0}\frac{\mu_j(\varepsilon)}{\varepsilon}=\mu_{j-2}, \qquad j\geqslant 2;\\ \lim_{\varepsilon\to 0}\mu_0(\varepsilon)=\min(\overline{\mu},\overline{\overline{\mu}}), \qquad \lim_{\varepsilon\to 0}\mu_1(\varepsilon)=\max(\overline{\mu},\overline{\overline{\mu}}),\end{gathered} \tag{4.28}$$

where $\overline{\mu}$, $\overline{\overline{\mu}}$ are the constants in (4.23), (4.27).

4.2. Conclusion. It is interesting to mention that the results obtained above for the spectral problem (4.1) can also be used to analyze boundary-value problems of the form

$$\begin{gathered}\varepsilon\,\frac{d^2y}{dx^2}+p(x)\,\frac{dy}{dx}+(q(x)+\lambda)y=0,\\ \biggl(\cos\theta_1-\frac{p(a)}{2}\sin\theta_1\biggr)y|_{x=a}-\varepsilon\sin\theta_1\,\frac{dy}{dx}\bigg|_{x=a}=0,\\ \biggl(\cos\theta_2-\frac{p(b)}{2}\sin\theta_2\biggr)y|_{x=b}-\varepsilon\sin\theta_2\,\frac{dy}{dx}\bigg|_{x=b}=0.\end{gathered} \tag{4.29}$$

Here $0<\varepsilon\ll 1$, $0\leqslant\theta_1<\pi$, $0<\theta_2\leqslant\pi$, $p(x),q(x)\in C^\infty([a,b])$ and, most importantly, the coefficient $p(x)$ has finitely many simple roots $x_1<x_2<\dots<x_n$ on the interval (a,b) and is non-zero at the other points of $[a,b]$. These zeros are commonly referred to as turning points.

By making the changes of variables

$$y=u\exp\biggl[-\frac{1}{2\varepsilon}\int_a^x p(s)\,ds\biggr], \qquad \mu=\varepsilon\lambda$$

in (4.29), we obtain the spectral problem (4.1) with potential

$$q(x,\varepsilon)=\frac{p^2(x)}{4}-\varepsilon\biggl(q(x)-\frac{p'(x)}{2}\biggr), \tag{4.30}$$

which satisfies Condition 4.1. Notice that the numbers (4.3) take the following form in the case (4.30):

$$\mu_{k,i} = i|p'(x_k)| + \frac{1}{2}\bigl(p'(x_k) + |p'(x_k)|\bigr) - q(x_k), \qquad k = 1, \dots, n, \quad i \geqslant 0. \tag{4.31}$$

As above, we assume that they are pairwise distinct.

We denote the eigenvalues of the spectral problem (4.29) by

$$\lambda_0(\varepsilon) < \lambda_1(\varepsilon) < \cdots < \lambda_j(\varepsilon) < \cdots \tag{4.32}$$

and let λ_j, $j \geqslant 0$, be the numbers (4.31) arranged in increasing order. Theorem 4.1 yields the following assertion.

Theorem 4.3. *Suppose that the constants θ_1, θ_2 in* (4.29) *satisfy the following restrictions* (*analogues of* (4.5)):

$$0 \leqslant \theta_1 < \frac{\pi}{2} + \arctan\biggl(\frac{|p(a)|}{2}\biggr), \qquad \frac{\pi}{2} - \arctan\biggl(\frac{|p(b)|}{2}\biggr) < \theta_2 \leqslant \pi. \tag{4.33}$$

Then, for every fixed $j \geqslant 0$, we have the following limiting equality for the jth eigenvalue (4.32):

$$\lim_{\varepsilon \to 0} \lambda_j(\varepsilon) = \lambda_j. \tag{4.34}$$

In the case when at least one of the conditions (4.33) does not hold, we can also use the corresponding results in § 4.1. For example, suppose that the following analogues of (4.20) hold:

$$\frac{\pi}{2} + \arctan\biggl(\frac{|p(a)|}{2}\biggr) < \theta_1 < \pi, \qquad \frac{\pi}{2} - \arctan\biggl(\frac{|p(b)|}{2}\biggr) < \theta_2 \leqslant \pi.$$

Then Theorem 4.2 and (4.23), (4.24) yield that

$$\lim_{\varepsilon \to 0} \lambda_j(\varepsilon) = \lambda_{j-1}, \quad j \geqslant 1; \qquad \lim_{\varepsilon \to 0} \lambda_0(\varepsilon) = -\infty. \tag{4.35}$$

Clearly, the equalities (4.35) also hold under the following conditions (analogues of (4.25)):

$$0 \leqslant \theta_1 < \frac{\pi}{2} + \arctan\biggl(\frac{|p(a)|}{2}\biggr), \qquad 0 < \theta_2 < \frac{\pi}{2} - \arctan\biggl(\frac{|p(b)|}{2}\biggr).$$

But if

$$\frac{\pi}{2} + \arctan\biggl(\frac{|p(a)|}{2}\biggr) < \theta_1 < \pi, \qquad 0 < \theta_2 < \frac{\pi}{2} - \arctan\biggl(\frac{|p(b)|}{2}\biggr),$$

then we obtain from (4.28) that

$$\lim_{\varepsilon \to 0} \lambda_j(\varepsilon) = \lambda_{j-2}, \quad j \geqslant 2; \qquad \lim_{\varepsilon \to 0} \lambda_0(\varepsilon) = \lim_{\varepsilon \to 0} \lambda_1(\varepsilon) = -\infty.$$

To conclude, we mention that the equalities (4.34) are well known in the case when $\theta_1 = 0$ and $\theta_2 = \pi$, that is, for the boundary conditions $y\big|_{x=a} = y\big|_{x=b} = 0$.

The corresponding fact was established in [22] for $j = 0$ and in a series of publications [23]–[26] for an arbitrary $j \geqslant 0$. These equalities were obtained there by studying the oscillation properties of solutions of the equation in (4.29). The analysis performed above provides another way to prove them and enables us to include this result in the context of the general theory of relaxation oscillations.

Bibliography

[1] L. S. Pontryagin and E. F. Mishchenko, "Periodic solutions of systems of differential equations which are close to discontinuous solutions", in the paper "The jubilee scientific session in the Faculty of Mechanics and Mathematics of Moscow State University", *Uspekhi Mat. Nauk* **10**:3(65) (1955), 193. (Russian)

[2] L. S. Pontryagin and E. F. Mishchenko, "Periodic solutions of systems of differential equations which are close to discontinuous solutions", *Dokl. Akad. Nauk SSSR* **102**:5 (1955), 889–891. (Russian)

[3] L. S. Pontryagin, "Systems of ordinary differential equations with a small parameter in the higher derivatives", *Proc. Third All-Union Mathematical Congress*, vol. 3, Academy of Sciences of the USSR, Moscow 1958, pp. 570–577. (Russian)

[4] L. S. Pontryagin, "Asymptotic behavior of the solutions of systems of differential equations with a small parameter in the higher derivatives", *Izv. Akad. Nauk SSSR Ser. Mat.* **21**:5 (1957), 605–626; English transl., Amer. Math. Soc. Transl. (2), vol. 18, Amer. Math. Soc., Providence, RI 1961, pp. 295–319.

[5] E. F. Mishchenko, "Asymptotic calculation of periodic solutions of systems of differential equations containing small parameters in the derivatives", *Izv. Akad. Nauk SSSR Ser. Mat.* **21**:5 (1957), 627–654; English transl., Amer. Math. Soc. Transl. (2), vol. 18, Amer. Math. Soc., Providence, RI 1961, pp. 199–230.

[6] E. F. Mishchenko and N. Kh. Rozov, *Differential equations with small parameters and relaxation oscillations*, Nauka, Moscow 1975; English transl., Math. Concepts Methods Sci. Engrg., vol. 13, Plenum Press, New York 1980.

[7] E. F. Mishchenko, Yu. S. Kolesov, A. Yu. Kolesov, and N. Kh. Rozov, *Asymptotic methods in singularly perturbed systems*, Fizmatlit, Moscow 1995; English transl., Monogr. Contemp. Math., Consultants Bureau, New York 1994.

[8] A. K. Zvonkin and M. A. Shubin, "Non-standard analysis and singular perturbations of ordinary differential equations", *Uspekhi Mat. Nauk* **39**:2(236) (1984), 77–127; English transl., *Russian Math. Surveys* **39**:2 (1984), 69–131.

[9] W. Eckhaus, "Relaxation oscillations including a standard chase on French ducks", *Asymptotic analysis* II, Lecture Notes in Math., vol. 985, Springer, Berlin 1983, pp. 449–494.

[10] V. A. Sobolev and E. A. Shchepakina, "Duck trajectories in a problem of combustion theory", *Differ. Uravneniya* **32**:9 (1996), 1175–1184; English transl., *Differential Equations* **32**:9 (1996), 1177–1186.

[11] V. A. Sobolev and E. A. Shchepakina, "Integral surfaces with change of stability and canard trajectories", *Izv. RAEN Ser. MMMIU* **1**:3 (1997), 176–187. (Russian)

[12] M. Diener, "Nessie et les canards", Publ. IRMA, vol. 76, IRMA, Strasbourg 1979.

[13] M. Diener, *Etude générique des canards*, Thése, IRMA, Strasbourg 1981.

[14] A. Yu. Kolesov, E. F. Mishchenko, and N. Kh. Rozov, "Solution to singularly perturbed boundary-value problems by the duck hunting method", *Algebra. Topology. Differential equations and their applications*, Collection of papers. On the 90th birthday of Academician Lev Semenovich Pontryagin, Tr. Mat. Inst. Steklova,

vol. 224, Nauka, Moscow 1999, pp. 187–207; English transl., *Proc. Steklov Inst. Math.* **224** (1999), 169–188.

[15] V. F. Butuzov, N. N. Nefedov, and K. R. Schneider, "Singularly perturbed problems in case of exchange of stabilities", *J. Math. Sci.* (*N. Y.*) **121**:1 (2004), 1973–2079.

[16] A. M. Krasnosel'skii, E. O'Grady, A. V. Pokrovskii, and D. I. Rachinskii, "Periodic canard trajectories with multiple segments following the unstable part of critical manifold", *Discrete Contin. Dyn. Syst. Ser. B* **18**:2 (2013), 467–482.

[17] E. Shchepakina, V. Sobolev, and M. P. Mortelli, *Singular perturbations. Introduction to system order reduction methods with applications*, Lecture Notes in Math., vol. 2114, Springer, Cham 2014.

[18] M. M. Vainberg and V. A. Trenogin, *Theory of branching of solutions of non-linear equations*, Nauka, Moscow 1969; English transl., Monogr. Textbooks Pure Appl. Math., Noordhoff, Leyden 1974.

[19] A. B. Vasil'eva and V. F. Butuzov, *Asymptotic expansions of solutions of singularly perturbed equations*, Nauka, Moscow 1973. (Russian)

[20] M. Abramowitz and I. A. Stegun (eds.), *Handbook of mathematical functions*, National Bureau of Standards Applied Mathematics Series, vol. 55, Superintendent of Documents, U.S. Government Printing Office, Washington, DC 1964; Russian transl., Nauka, Moscow 1979.

[21] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York–Toronto–London 1955; Russian transl., Inostr. Lit., Moscow 1958.

[22] Yu. S. Kolesov and V. F. Chaplygin, "The nonoscillation of the solutions of singularly perturbed equations of second order", *Dokl. Akad. Nauk SSSR* **199**:6 (1970), 1240–1242; English transl., *Soviet Math. Dokl.* **12** (1971), 1268–1271.

[23] S. A. Kashchenko, "Limiting values of the eigenvalues of the first boundary-value problem for a singularly perturbed second-order differential equations with turning points", Vestn. Yarosl. Gos. Univ., vol. 10, Yaroslavl' Univ. Press, Yaroslavl' 1974, pp. 3–39. (Russian)

[24] S. A. Kashchenko, "Asymptotic behaviour of the eigenvalues of the first boundary-value problem for a singularly perturbed second-order differential equations with turning points", Vestn. Yarosl. Gos. Univ., vol. 10, Yaroslavl' Univ. Press, Yaroslavl' 1974, pp. 40–64. (Russian)

[25] S. A. Kashchenko, *Stability of second-order equations with periodic coefficients*, a tutorial, Yaroslavl' Univ. Press, Yaroslavl' 2005. (Russian)

[26] S. A. Kashchenko, "Asymptotic behaviour of the eigenvalues of the first boundary-value problem for a singularly perturbed second-order differential equations with turning points", *Model. i Analiz Inform. Sistem* **22**:5 (2015), 682–710. (Russian)

Sergey D. Glyzin
Yaroslavl' State University
Scientific Centre of RAS in Chernogolovka
E-mail: glyzin@uniyar.ac.ru

Andrei Yu. Kolesov
Yaroslavl' State University
E-mail: kolesov@uniyar.ac.ru

Nikolai Kh. Rozov
Moscow State University
E-mail: fpo.mgu@mail.ru

Received 10/MAR/16
Translated by A. V. DOMRIN