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Gravitational Interaction of Fermion Antisymmetric Tensor Fields

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Abstract. We investigate the coupling of classical and quantum antisymmetric tensor fields, which describe fermions, with the gauge gravitational field. We show that within the framework of the classical Einstein-Cartan theory the new generalized nonlinear fermion theory can be formulated, which turns out to be the correct microscopic description of the Weyssenhoff spinning fluid. The one-loop gravitational counterterms and the conformal stress tensor and the axial vector current anomalies are obtained. The differences between the antisymmetric tensor fermions and the usual Dirac spinor fields are discussed.

Gravitative Wechselwirkung von Fermischen antisymmetrischen Tensorfeldern

Inhaltsübersicht. Wir untersuchen die Kopplung von klassischen und quantisierten antisymmetrischen Tensorfeldern, die Fermionen beschreiben, mittels eines gravitativen Eichfeldes. Wir zeigen, daß innerhalb der klassischen Einstein-Cartan-Theorie die neue allgemeine nichtlineare Fermionentheorie formuliert werden kann. Sie erweist sich als die korrekte mikroskopische Beschreibung der Weyssenhoffschen Spinflüssigkeit. Man erhält die gravitativen 1-Schleifen-Gegenterme, den konformen Spannungstensor und die achsiale Vektorstromanomalie. Die Unterschiede zwischen den antisymmetrischen Tensorfermionen und den üblichen Diracschen Spinorfeldern werden diskutiert.

1. Introduction

In 1928 Dirac [1] has established the relativistic wave equation which describes a particle with spin 1/2 in terms of spinors. Independently Ivanenko and Landau [2] have suggested an alternative relativistic equation for the wave function represented by a system of antisymmetric tensor fields. Recently the mathematician E. Kähler [3] has rediscovered this equation and studied its properties in detail; in the modern literature the mentioned equation is often refered to as the Dirac-Kähler equation.

During the last years much attention is paid to the investigation of field-theoretic models with antisymmetric tensor fields (ATF). It turns out that ATF play an important role in the dual string model, in supergravity, in the gauge theory of gravity. In quantum chromodynamics they are realized as asymptotic fields for the bound configurations and are necessary for the explanation of the $U(1)$ problem in QCD (see e.g. [4, 5]). ATF represent a new type of the gauge invariance [5] and, what is most important, they can be described with the help of differential-geometric methods as exterior differential forms on a space-time manifold.

In all above mentioned models ATF describe boson fields, i.e. the particles with an integer spin. This is natural from the point of view of the tensorial character of their transformation law under the action of the Lorentz group. At the same time there exists the fundamental connection between the Clifford algebra and the exterior form algebra [6], and one can suppose the possibility of the description of half-integer spin fields by means of ATF. In the refs. [2, 3] this supposition is supported and the concept of the fermions without spinors is introduced. This notion has proved to be very useful in different fieldtheoretical models. We want to mention first, that already in the earlier works [3, 8] it was shown that ATF-fermions behave exactly as the usual Dirac spinor fields when interacting with an electromagnetic field. However they have some advantages, since unlike the Dirac fields, which one cannot unambigiously transfer on the lattice (cf. [14]), ATF-fermions can be naturally described on the lattice within the framework of the homology theory on a space-time [9]. Secondly, it is very tempting to construct a supersymmetric generalization of this model, since then both fermions and bosons will be unified in a single geometrical object-differential form (for preliminary discussion of this possibility see [11]). An important aspect is also the natural account of the topological properties of space-time and their influence on physical processes. Finally, it is worth mentioning that one does not need the tetrad fields for the description of fermions on a Riemannian background, since the exterior form formalism is by definition coordinate invariant. As a consequence the spin-structure can be defined globally over an arbitrary space-time contrary to the case of the Dirac fields.

In the present paper we study the differences of the gravitational interactions of ATF-fermions as compared to the usual results of Dirac fields in a curved space-time.

2. ATF-Fermions in Flat Space-Time

In this section we brielfy review the description of fermions without Dirac fields in the flat Minkowsky space-time.

Let Φ be the complex inhomogeneous differential form on the four-dimensional differentiable manifold,

$$\Phi = \sum_{k=0}^{4} \frac{1}{k!} \varphi_{\mu_1 \dots \mu_k} \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}. \tag{1}$$

The fermion Ivanenko-Landau-Kähler equation is

$$\{i(d - \delta) - m\} \, \Phi = 0, \tag{2}$$

where d and δ are respectively the exterior differential and co-differential: for the form

$$\psi = \frac{1}{k!} \psi_{\mu_1 \dots \mu_k} \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k},$$

$$d\psi = \frac{1}{k!} (\partial_{\mu_1} \psi_{\mu_2 \dots \mu_{k+1}}) \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{k+1}},$$

$$\delta\psi = -\frac{1}{(k-1)!} (\partial^\alpha \psi_{\alpha\mu_1 \dots \mu_{k-1}}) \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{k-1}}.$$

Despite the compact form of the ILK equation (2) in differential forms it will be more convenient for us to write it down in components. We then describe the inhomogeneous field Φ by the set of ATF of the rank $0, \dots, 4$,

$$\Phi = \{\varphi_{\mu_1 \dots \mu_k}, \, k = 0, 1, \dots, 4\}, \tag{3}$$

for which the eq. (2) takes the following form

$$i(k \, \partial_{[\mu_1} \varphi_{\mu_2 \dots \mu_k]} + \partial^\alpha \varphi_{\alpha\mu_1 \dots \mu_k}) - m\varphi_{\mu_1 \dots \mu_k} = 0, \tag{4}$$

$$k = 0, 1, \dots, 4.$$

The set of equations (4) can be derived from the action principle with the Lagrangian

$$L = \sum_{k=0}^{4} \frac{1}{k!} \Big\{ \frac{i}{2} \bar{\varphi}^{\mu_1 \dots \mu_k} (k\, \partial_{\mu_1} \varphi_{\mu_2 \dots \mu_k} + \partial^\alpha \varphi_{\alpha \mu_1 \dots \mu_k}) - \frac{i}{2} (k\, \partial_{\mu_1} \bar{\varphi}_{\mu_2 \dots \mu_k} + \partial^\alpha \bar{\varphi}_{\alpha \mu_1 \dots \mu_k}) \varphi^{\mu_1 \dots \mu_k} - m \bar{\varphi}_{\mu_1 \dots \mu_k} \varphi^{\mu_1 \dots \mu_k} \Big\}. \tag{5}$$

As it was shown earlier [9] there exists a transformation, we call it the canonical one, from antisymmetric tensors to the Dirac spinors. It can be formulated as follows

$$\varphi_{\mu_1 \dots \mu_k} = (-1)^{\frac{k(k-1)}{2}} Tr(\psi \Gamma_{\mu_1 \dots \mu_k}). \tag{6}$$

Here ψ is the Dirac second order spinor, i.e. it can be represented by a complex 4×4 matrix ψ_{ij} with $i, j = 1, \dots, 4$ as the spinor indices.

The Dirac-spin tensors

$$\Gamma_{\mu_1 \dots \mu_k} = \gamma_{[\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_k]}, \quad k = 1, \dots, 4,$$

together with the identity matrix form the basis of the fourdimensional Dirac algebra, defined by the standard relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} I.$$

Note that for the signature of the Minkowski metric and for the Dirac γ-matrices we use the Bogoliubov-Shirkov conventions. The complex conjugation of (6) defines the conjugated Dirac two-spinor via

$$\bar{\varphi}_{\mu_1 \dots \mu_k} = Tr(\bar{\psi} \Gamma_{\mu_1 \dots \mu_k}), \tag{7}$$

where

$$\bar{\psi} = \gamma^0 \psi^+ \gamma^0.$$

Substituting (6)—(7) into (5) we find for the Lagrangian

$$L = 4Tr \left\{ \frac{i}{2} (\bar{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi) - m \bar{\psi} \psi \right\}. \tag{8}$$

This result is easily obtained with the help of the main Fierz identity

$$\delta_{im} \delta_{jn} = \frac{1}{4} \sum_{k=0}^{4} \frac{1}{k!} (-1)^{\frac{k(k-1)}{2}} (\Gamma_{\mu_1 \dots \mu_k})_{nm} (\Gamma^{\mu_1 \dots \mu_k})_{ij}.$$

In the following we also use the identities for the products of the Dirac matrices,

$$\gamma^\nu \Gamma^{\mu_1 \dots \mu_k} = \Gamma^{\nu \mu_1 \dots \mu_k} + k \eta^{\nu [\mu_1} \Gamma^{\mu_2 \dots \mu_k]}, \tag{9a}$$

$$\Gamma^{\mu_1 \dots \mu_k} \gamma^\nu = \Gamma^{\mu_1 \dots \mu_k \nu} + k \Gamma^{[\mu_1 \dots \mu_{k-1}} \eta^{\mu_k] \nu}. \tag{9b}$$

Let us briefly discuss the invariance properties of the theory under consideration. Under the Lorentz group transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu,$$

$$\varphi^{\mu_1 \dots \mu_k} \to \varphi'^{\mu_1 \dots \mu_k} = \Lambda^{\mu_1}{}_{\nu_1} \dots \Lambda^{\mu_k}{}_{\nu_k} \varphi^{\nu_1 \dots \nu_k}, \tag{10}$$

the two-spinors ψ_{ij} transform as usual,

$$\psi \to \psi' = S \psi S^{-1} \tag{11}$$

where the matrix S realizes the spinor representation of the Lorentz group and is connected with Λ via

$$S^{-1} \gamma^\mu S = \Lambda^\mu_\nu \gamma^\nu.$$

The invariance of (8) under the Lorentz group yields the conservation of the spin, which includes contributions of the standard spin of the Dirac field (the left-spin in the terminology of the ref. [7]) and the additional right-spin. The latter corresponds to the Lorentz subgroup of the group of right transformations

$$\psi \to \psi' = \psi M,\ \overline{\psi} \to \overline{\psi}' = M^{-1}\overline{\psi}, \tag{12}$$

which evidently leave the Lagrangian (8) invariant. As one can easily see the matrix M realizes the representation of the conformal group $SO(2, 4)$. This fact was noticed earlier by a number of authors [15] who have investigated various forms of the two-spinor equation, derived from (8), and/or the equation (4). However all of them related the equation under consideration with bosons, and not with fermions, as it is supposed in the Ivanenko-Landau-Kähler approach. One can show that quantisation of the theory (8) according to the Bose-Einstein statistics, as proposed in [15], leads to inconsistencies and the correct way out is to adopt the Fermi-Dirac quantisation rules [16], thus interpreting the field (3) as the set of four Dirac fields with the spin 1/2. These Dirac fields are presicely those minimal left ideals, introduced in [3, 8] and in the flat space-time in the presence of electromagnetic field they decouple from each other and behave as four independent fermion particles with identical masses. However in a curved space-time this is not the case and a separate study of the gravitational interactions of ATF-fermions is required.

3. Gravitational Interaction of Classical ATF-Fermions

In the framework of the gauge theory of gravity [17] the geometrical structure of space-time is determined by the gauge gravitational potentials, which in the case of the Poincaré group are the tetrad fields and the local Lorentz connection,

$$h^a_\mu,\ \tilde{\Gamma}^{ab}{}_\mu = -\tilde{\Gamma}^{ba}{}_\mu.$$

These define the Riemann-Cartan structure which consists of the metric $g_{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab}$ and the global non-symmetric connection $\tilde{\Gamma}^\alpha_{\beta\mu} = h^\alpha_a h^b_\beta \tilde{\Gamma}^a_{b\mu} + h^\alpha_a\, \partial_\mu h^a_\beta$, compatible with the metric, $\tilde{\nabla}_\alpha g_{\mu\nu} = 0$, but possessing the non-zero torsion $Q^\alpha_{\cdot\mu\nu} = \tilde{\Gamma}^\alpha_{[\mu\nu]}$. Hereinafter we denote by a tilde the objects, constructed from the Riemann-Cartan connection with torsion, reserving the usual notation without additional marks for the Riemannian objects. constructed from the Christoffel symbols, $\{^\alpha_{\beta\mu}\} = \frac{1}{2} g^{\alpha\nu}(\partial_\beta g_{\mu\nu} + \partial_\mu g_{\beta\nu} - \partial_\nu g_{\beta\mu})$.

Let us now introduce the interaction of the classical ATF-fermion field with the gauge gravitational field. As usual (see e.g. [18]) we assume the minimal coupling recipe is valid according to which the Minkowsky metric is substituted by the Riemannian one and all the partial derivatives are replaced by the Riemann-Cartan covariant ones,

$$\partial_{[\mu_1}\varphi_{\mu_2 \ldots \mu_k]} \to \tilde{\nabla}_{[\mu_1}\varphi_{\mu_2\ldots\mu_k]},\ \partial^\alpha \varphi_{\alpha\mu_1\ldots\mu_k} \to \tilde{\nabla}^\alpha \varphi_{\alpha\mu_1\ldots\mu_k}.$$

Hence in the two-spinor variables we find

$$\begin{aligned}\tilde{\nabla}_{[\mu_1}\varphi_{\mu_2\ldots\mu_k]} &= (-1)^{\frac{(k-1)(k-2)}{2}} Tr(\tilde{\nabla}_{[\mu_1}\psi \Gamma_{\mu_2\ldots\mu_k]}) \\ &= (-1)^{\frac{k(k-1)}{2}} Tr(\Gamma_{[\mu_1\ldots\mu_{k-1}} \tilde{\nabla}_{\mu_k]}\psi),\end{aligned} \tag{13}$$

where

$$\tilde{\nabla}_\mu \psi = \partial_\mu \psi + [\tilde{\Gamma}_\mu, \psi]$$

is the spinor covariant derivative, defined by the spinor connection

$$\tilde{\Gamma}_\mu = -\frac{1}{4}\gamma^a\gamma_b\tilde{\Gamma}^b_{a\mu} = -\frac{1}{4}\sigma_{ab}\tilde{\Gamma}^{ab}{}_\mu. \tag{14}$$

As usually the latter is defined so that the curved γ-matrices are covariantly constant, $\tilde{\nabla}_\mu\gamma_\nu = 0$, and the spin-curvature is

$$\hat{R}_{\mu\nu} = \partial_\mu\tilde{\Gamma}_\nu - \partial_\nu\tilde{\Gamma}_\mu + [\tilde{\Gamma}_\mu\tilde{\Gamma}_\nu] = \frac{1}{4}\sigma_{ab}\tilde{R}^{ab}{}_{\mu\nu}. \tag{15}$$

Analogously

$$\begin{aligned}\tilde{\nabla}^\alpha\varphi_{\alpha\mu_1\ldots\mu_k} &= (-1)^{\frac{k(k+1)}{2}}\,Tr(\tilde{\nabla}^\alpha\psi\Gamma_{\alpha\mu_1\ldots\mu_k})\\ &= (-1)^{\frac{k(k-1)}{2}}\,Tr\{\Gamma_{\mu_1\ldots\mu_k\alpha}\tilde{\nabla}^\alpha\psi\},\end{aligned} \tag{16}$$

and thus finally, making use of (9), we get

$$k\,\tilde{\nabla}_{[\mu_1}\varphi_{\mu_2\ldots\mu_k]} + \tilde{\nabla}^\alpha\varphi_{\alpha\mu_1\ldots\mu_k} = (-1)^{\frac{k(k-1)}{2}}\,Tr(\Gamma_{\mu_1\ldots\mu_k}\gamma^\alpha\tilde{\nabla}_\alpha\psi). \tag{17}$$

Thus the Lagrangian of the gravitationally interacted ATF-fermion theory is the following,

$$L = 4Tr\left\{\frac{i}{2}(\bar{\psi}\gamma^\mu\tilde{\nabla}_\mu\psi - \tilde{\nabla}_\mu\bar{\psi}\gamma^\mu\psi) - m\bar{\psi}\psi\right\}. \tag{18}$$

The field equations are then

$$\{i\gamma^\mu(\tilde{\nabla}_\mu - Q_\mu) - m\}\,\psi = 0. \tag{19}$$

It is important to note that the group of symmetries of the theory under consideration is changed in the curved space-time. Indeed, since in general the generators of $SO(2, 4)$ do not commute with the Lorentz generators, the theory (18), (19) is not invariant under the right transformations. The only exception is the case of the right γ_5-transformations, under which (18) is invariant both in massless and in massive cases as well as in a curved space-time.

The theory (18) in the massless case ($m = 0$) is invariant also under the local conformal transformations, $g_{\mu\nu} \to e^{2\sigma}g_{\mu\nu}$, $\psi \to e^{3\sigma/2}\psi$, where $\sigma = \sigma(x)$. However the Weyl transformations of the metric must be interpreted in the sense of the tetrad scaling, which defines the conformal transformations in the Riemann-Cartan space-time according to the refs. [19]. As for the torsion-free case of the Riemannian theory, it was shown in [20] that the theory of ATF-fermions is invariant only under the global conformal transformations.

Let us now consider the self-consistent theory of the ATF-fermions, interacting with the gravitational field in the framework of the Einstein-Cartan theory (ECT). The total Lagrangian is then

$$L = -\frac{1}{2\varkappa}\tilde{R} + Tr\left\{\frac{i}{2}(\bar{\psi}\gamma^\mu\tilde{\nabla}_\mu\psi - \tilde{\nabla}_\mu\bar{\psi}\gamma^\mu\psi) - m\bar{\psi}\psi\right\} \tag{20}$$

where we denoted as usual $\varkappa = \dfrac{8\pi G}{c^4}$.

The gravitational field equations are derived from the variation of the action (20) by h^a_μ and $\tilde{\Gamma}^a_{b\mu}$. The corresponding Einstein-Palatini equations are as follows

$$\tilde{R}^a_\mu - \frac{1}{2} h^a_\mu \tilde{R} = \frac{i\varkappa}{2} Tr(\bar{\psi}\gamma^a \tilde{\nabla}_\mu \psi - \tilde{\nabla}_\mu \bar{\psi}\gamma^a \psi), \tag{21}$$

$$Q^\alpha_{\cdot\mu\nu} + \delta^\alpha_\mu Q_\nu - \delta^\alpha_\nu Q_\mu = \frac{\varkappa}{8} S^\alpha_{\mu\nu} = \frac{i\varkappa}{8} Tr[\bar{\psi}(\{\gamma^\alpha, \sigma_{\mu\nu}\} \psi - 2\sigma_{\mu\nu}\bar{\psi}\gamma^\alpha \psi]. \tag{22}$$

The Einstein equations (21) turn out to be the same as if the gravitational field interacts with four fermion Dirac fields, and this at the first sight supports the earlier results in a flat space-time. However the Palatini equation (22) differs considerably from the expected analogy with the four types of fermions in a Riemann-Cartan space; here we obtain, that not only the usual (left one) spin, but also the right-spin produce the space-time torsion via (22). From this equation one easily gets

$$Q^\alpha_{\mu\nu} = -\frac{i\varkappa}{4} \{-\varepsilon^\alpha_{\cdot\mu\nu\beta} J^\beta_5 + Tr[[\sigma_{\mu\nu}\, \delta^\alpha_\beta + \delta^\alpha_{[\mu}\sigma_{\nu]\beta}]\, \bar{\psi}\gamma^\beta \psi]\}, \tag{23}$$

where $-J^\mu_5 = i\, Tr(\bar{\psi}\gamma^\mu\gamma_5\psi)$ is the standard axial current, connected with the left-spin, and the rest is the right-spin contribution. Note that in this theory unlike the usual ECT with Dirac fields the trace of torsion does not vanish,

$$Q_\mu = Q^\nu_{\mu\nu} = \frac{i\varkappa}{8} Tr(\sigma_{\mu\nu}\bar{\psi}\gamma^\nu \psi). \tag{24}$$

Substituting (23) back into (19) we obtain the non-linear spinor equation, which generalizes the fundamental ψ^4-equation [21]. Interpreting the two-spin ψ_{ij} as the set of four spin 1/2 fermion fields—as proposed in [12] we can think of them as the four generations of quarks—we can now suggest the obtained non-linear spinor equation to underly the generalized sub-quark theory with non-trivial selfinteraction between the different types of fermion constituents. Having in mind that the geometrical arguments often lead to the correct mathematical schemes of the physical theories (as is the case for example for the standard non-linear spinor theory [22]), one can suppose that thus obtained generalized non-linear equation will be useful in the study of possible sub-quark models.

Instead of writing down this equation explicitly it appears more interesting to give the corresponding effective nonlinearity in the Lagrangian. Since the Riemann-Cartan curvature scalar can be decomposed into the Riemannian and torsion-dependent parts,

$$\tilde{R} = R - 4Q_\mu Q^\mu - K_{\mu\nu\lambda}K^{\nu\lambda\mu},$$

and the contorsion

$$K^\alpha_{\mu\nu} = Q^\alpha_{\mu\nu} + Q_{\mu\nu,}{}^\alpha + Q_{\nu\mu,}{}^\alpha,$$

is easily expressed in terms of spinor fields via (23), we get

$$\begin{aligned} L = &-\frac{1}{2\varkappa} R + Tr\left\{\frac{i}{2}(\bar{\psi}\gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi}\gamma^\mu \psi) - m\bar{\psi}\psi\right\} \\ &+ \frac{1}{2\varkappa}(4Q_\mu Q^\mu + K_{\mu\nu\lambda}K^{\nu\lambda\mu}) + \frac{1}{8} K^{\nu\lambda\mu} S_{\mu\nu\lambda} \end{aligned} \tag{25}$$

$$= -\frac{1}{2\varkappa} R + Tr\left\{\frac{i}{2}(\bar{\psi}\gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi}\gamma^\mu \psi) - m\bar{\psi}\psi\right\} + L_{\text{eff}},$$

$$L_{\text{eff}} = \frac{\varkappa}{128}(S^{\lambda\mu\nu}S_{\lambda\mu\nu} + 2S^{\mu\lambda\nu}S_{\lambda\mu\nu} - 2S^\mu S_\mu).$$

Let us point out the interesting possibilities, offered by the generalized non-linear theory. Let us suppose the following structure of the two-spinor ψ_{ij}, which can occur in the concrete problem,

$$\psi_{ij} = \varphi_i \bar{\chi}_j, \tag{26}$$

where φ and χ are the usual Dirac spinors. Then, denoting their bilinear combinations

$$S_{\mu\nu} = i(\bar{\chi}\sigma_{\mu\nu}\chi), \quad S = \bar{\chi}\chi,$$
$$I_\mu = \bar{\varphi}\gamma_\mu\varphi, \quad J_\mu = \bar{\varphi}\gamma_\mu\gamma_5\varphi,$$

we get

$$S^\alpha_{\mu\nu} = 2(S\varepsilon^\alpha_{\beta\mu\nu}J^\beta - I^\alpha S_{\mu\nu}). \tag{27}$$

Hence the effective non-linear Lagrangian reduces to

$$L_{\text{eff}} = (S^{\alpha\mu\nu}S_{\alpha\mu\nu} + 2S_{\alpha\mu\nu}S^{\mu\alpha\nu} - 2S_\mu S^\mu)\frac{\varkappa}{128} = \frac{\varkappa}{32}(S\varepsilon_{\alpha\beta\mu\nu}J^\beta + S_{\mu\nu}I_\alpha)^2. \tag{28}$$

This result has two far-going consequences. First, as we see from (27) the ATF-fermions turn out to be the correct microscopic description of the spinning classical matter, which in the fluid like limit is represented by the well-known WEYSSENHOFF-RAABE [23] fluid. Recently it has been recognized that the usual Dirac matter yields a different quasi-classical description (cf. e.g. [24]) and hence the problem has arisen, what matter can underly the Weysenhoff's approximation. Now we can suggest a possible answer: the Weysenhoff-Raabe spinning fluid is naturally obtained from the ATF-fermion matter, and the corresponding spin density is determined by the "internal" right-spin fermions, $S_{\mu\nu} = S_{\mu\nu}(\chi)$, while the current I^μ vector, proportional to the fluid four-velocity u^μ is defined by the usual spinor field φ.

The second consequence is related to the above observation and consists in the prediction of the possibility to avert the singularity in the solution of the Einstein equations (21). Indeed, although the first term in (28) is strictly negative and hence it describes the attraction of fermions and accelerates the cosmological collapse, but the second term is positive for the polarised ATF-fermions and thus the final answer about the singularity depends on the difference between these two terms. Anyhow the possibility of averting the singularity, absent in the case of standard spinor matter, is non-zero for the fermion matter under consideration.

Note that for neutrino, when $\varphi = \pm\gamma_5\varphi$, we again obtain the theory without any non-linearity, like in the Dirac neutrino case.

4. Quantised ATF-Fermions in Curved Space-Time

Let us investigate now the quantised ATF theory in the classical Riemann-Cartan space-time. The most important quantities are the one-loop gravitational counterterms, which determine the quasiclassical approximation for the Einstein-Cartan theory. In this section we shall calculate these counterterms and also the related conformal and axial current anomalies. Earlier we have obtained the analogous results for the standard Dirac spinor field in spaces with torsion [25]. Now we use the same formalism and notations for ATF-fermions, and for the details the reader should address the ref. [25].

The generalized Dirac operator, see (19),

$$\Delta = i\gamma^\mu(\tilde{\nabla}_\mu - Q_\mu) - m \equiv \Delta_{ij,kl} \tag{29}$$

formally coincides with that of the standard spinor field (cf. eq. (4.1) of [25]) but note however that the difference is in the covariant derivative: now it acts on the matrix

ψ_{ij} and hence the V-bundle connection ω_μ (see [25]) is

$$\omega_{\mu ij,kl} = (\Gamma_\mu)_{ik}\,\delta_{lj} - (\Gamma_\mu)_{lj}\,\delta_{ik}. \tag{30}$$

(Explicitly the covariant derivative is now $(\tilde{\nabla}_\mu \psi)_{ij} = \partial_\mu \psi_{ij} + \omega_{\mu ij,kl}\psi_{kl}$.)

The V-bundle curvature for the connection (30)

$$F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + \omega_\mu\omega_\nu - \omega_\nu\omega_\mu \equiv (F_{\mu\nu})_{ij,kl} = (\hat{R}_{\mu\nu})_{ik}\,\delta_{lj} - (\hat{R}_{\mu\nu})_{lj}\,\delta_{ik} =$$

in view of (15) becomes

$$= \frac{1}{4}\,\tilde{R}_{\alpha\beta\mu\nu}\{(\sigma^{\alpha\beta})_{ik}\,\delta_{lj} - (\sigma^{\alpha\beta})_{lj}\,\delta_{ik}\}. \tag{31}$$

The self-adjoint second order Dirac operator $D = \Delta^*\Delta$ for (29) is again formally (and notationally) the same as the operator (4.3), (4.4) of the ref. [25]. However taking into account the matrix nature of ψ_{ij} we get the explicit form of the matrices S^μ and X in

$$D = -(g_{\mu\nu}\nabla_\mu\nabla_\nu + 2S^\mu\nabla_\mu + X).$$

They are:

$$\left.\begin{aligned}
(S^\mu)_{ij,kl} &= \frac{1}{2}\,Q^\mu_{\alpha\beta}(\sigma^{\alpha\beta})_{ik}\,\delta_{lj},\\
(X)_{ij,kl} &= \left\{\frac{1}{2}(\sigma^{\alpha\beta}\hat{R}_{\alpha\beta})_{ik} - \nabla_\alpha Q_\beta(\sigma^{\alpha\beta})_{ik} + (X_0)_{ik}\right\}\delta_{lj}\\
&\quad - \frac{1}{2}(\sigma^{\alpha\beta})_{ik}\,(\hat{R}_{\alpha\beta})_{lj}
\end{aligned}\right\} \tag{32}$$

where

$$(X_0)_{ik} = (-\tilde{\nabla}_\mu Q^\mu + Q_\mu Q^\mu + m^2)\,\delta_{ik}.$$

From these for the matrix Z we get

$$(Z_{ij,kl}) = (Z^{(1)})_{ik}\,\delta_{lj} + (Z^{(2)})_{ij,kl}, \tag{33}$$

where

$$\begin{aligned}
(Z^{(1)})_{ik} &= \frac{1}{8}\,(\tilde{R}_{\alpha\beta\mu\nu} - 2Q^\varrho_{\alpha\beta}Q_{\varrho\mu\nu})\,(\sigma^{\alpha\beta}\sigma^{\mu\nu})_{ik}\\
&\quad + (X_0)_{ik} + \left(-\tilde{\nabla}_\alpha Q_\beta + Q^\mu_{\alpha\beta}Q_\mu - \frac{1}{2}\tilde{\nabla}_\mu Q^\mu_{\alpha\beta}\right)(\sigma^{\alpha\beta})_{ik},
\end{aligned} \tag{34}$$

$$(Z^{(2)})_{ij,kl} = -\frac{1}{8}\,\tilde{R}_{\mu\nu\alpha\beta}(\sigma^{\alpha\beta})_{ik}\,(\sigma^{\mu\nu})_{lj}. \tag{35}$$

Analogously for the matrix $Y_{\mu\nu}$ (see eq. (3.9) of [25]) one obtains

$$(Y_{\mu\nu})_{ij,kl} = (Y^{(1)}_{\mu\nu})_{ik}\,\delta_{lj} + (Y^{(2)}_{\mu\nu})_{lj}\,\delta_{ik}, \tag{36}$$

where

$$(Y^{(1)}_{\mu\nu})_{ik} = \frac{1}{4}\,\Phi_{\alpha\beta\mu\nu}(\sigma^{\alpha\beta})_{ik}, \tag{37}$$

$$(Y^{(2)}_{\mu\nu})_{lj} = -\frac{1}{4}\,\tilde{R}_{\alpha\beta\mu\nu}(\sigma^{\alpha\beta})_{lj}, \tag{38}$$

$$\Phi_{\alpha\beta\mu\nu} = \tilde{R}_{\alpha\beta\mu\nu} + 4\tilde{\nabla}_{[\mu}Q_{\nu]\alpha\beta} - 4Q^\varrho_{\mu\nu}Q_{\varrho\alpha\beta} - 8Q_{[\mu|\alpha\varrho|}Q^\varrho_{\nu]\beta}. \tag{39}$$

Thus we have

$$Tr(Z) = 4Tr(Z^{(1)}), \tag{40a}$$

$$Tr(Z)^2 = 4Tr(Z^{(1)})^2 + Tr(Z^{(2)})^2, \tag{40b}$$

$$Tr(Y_{\mu\nu} Y^{\mu\nu}) = 4Tr\{Y^{(1)}_{\mu\nu} Y^{(1)\mu\nu} + Y^{(2)}_{\mu\nu} Y^{(2)\mu\nu}\}, \tag{40c}$$

and hence the b_4 coefficient for the Minakshisundaram expansion of the heat kernel for the operator (29) is given by the expression

$$b_4(D) = 4b_4(D_{\text{DIRAC}}) + \Delta b_4, \tag{41}$$

where

$$\Delta b_4 = \frac{2}{3(4\pi)^2} \tilde{R}_{\varkappa\beta\mu\nu} \tilde{R}^{\varkappa\beta\mu\nu}.$$

We thus see that the Minakshisundaram coefficient (41) for the ATF-fermion field is different from a naive sum of contributions of four types of ordinary spinor fields (represented in (41) by the first term) by the Yang-Mills type curvature contraction. Consequently, in the quasiclassical approximation to the ECT with ATF-fermions the torsion (or better to say, the local Lorentz connection) becomes totally propagating, and not only the axial trace part of it. It is also very important that the additional gravitational counterterm in (41) has a simple Poincaré invariant structure $\tilde{R}^{ab}{}_{\cdot\cdot\mu\nu} \tilde{R}_{ab}{}^{\mu\nu}$, as compared to the usual spinors in U_4.

Let us now calculate the anomalies for the ATF-fermions.

Like in the ordinary theory, the massless case of the theory (18) is invariant under the left γ_5-transformations,

$$\psi \to \psi' = \gamma_5 \psi,$$

and the left axial (classical) current is conserved,

$$J^\mu_5 = i\, Tr(\bar{\psi}\gamma_5\gamma^\mu\psi), \quad \nabla_\mu J^\mu_5 = 0.$$

However, using the method of ref. [25] (sect. 4.3) one can easily get for the vacuum average

$$\langle \nabla_\mu J^\mu_5 \rangle = \frac{\delta}{\delta\alpha(x)} \ln \det \Delta(\alpha)|_{\alpha=0} = 2(4\pi)^{-2}\, Tr\, [(\gamma_5)_{ii'} \delta_{jj'} (K_4)_{i'j',kl}],$$

and after taking the trace we find

$$\langle \nabla_\mu J^\mu_5 \rangle = 4\langle \nabla_\mu J^\mu_5 \rangle_{\text{DIRAC}} + P, \tag{42}$$

where

$$P = -\frac{1}{32\pi^2} \tilde{R}_{\varrho\sigma\alpha\beta} \tilde{R}^{\varrho\sigma}{}_{\mu\nu} \varepsilon^{\mu\nu\alpha\beta}.$$

Again we see that in additions to the four times result for the Dirac case the nontrivial contribution of the Riemann-Cartan Pontriagin integrand is present.

In the previous section we have seen that the theory (18) is invariant in curved spacetime under the right γ_5-transformations, the only relict of the $SO(2, 4)$ right symmetry of the flat space model. Thus on the classical level the right axial current, $J^\mu_R = i\, Tr(\bar{\psi}\gamma^\mu\psi\gamma_5)$ is also conserved,

$$\nabla_\mu J^\mu_R = 0.$$

Let us show that in quantum theory this conservation law is not disturbed, i.e. there are no anomalies for it.

Indeed, let us find the vacuum average,

$$\langle \nabla_\mu J^\mu_R \rangle = \frac{\delta}{\delta\alpha(x)} \ln \det \check{\Delta}(\alpha)|_{\alpha=0} \quad (43)$$

where the operator $\check{\Delta}(\alpha)$ is defined as follows

$$(\check{\Delta})_{ij,kl} = \Delta_{ij,kl} - i(\gamma_5)_{lj} (\gamma^\mu)_{ik} \partial_\mu \alpha = \Delta_{ij,kl} + (\hat{\alpha} \Delta_{im,kl} - \Delta_{im,kl}\hat{\alpha}) (\gamma_5)_{mj}. \quad (44)$$

Notice however that the generalized Dirac operator (29) commutes with the right γ_5 multiplication,

$$\Delta_{im,kl}(\gamma_5)_{mj} = (\gamma_5)_{lm} \Delta_{ij,km}.$$

Hence

$$(\check{\Delta}^2)_{ij,kl} = \Delta^2_{ij,kl} + 2[\hat{\alpha} \Delta^2_{im,kl}(\gamma_5)_{mj}] + O(\alpha^2), \quad (45)$$

and for the arbitrary degree of $\check{\Delta}^2$ we then find

$$(\check{\Delta}^2)^n_{ij,kl} = (\Delta^2)^n_{ij,kl} + 2n[\hat{\alpha} (\Delta^2_{im,kl}) (\gamma_5)_{mj}] + f^{(n)}_{ij,kl}, \quad (46)$$

where

$f^{(n)}{}_{ij,kl}$ depend only linearly on $\hat{\alpha}^2, \hat{\alpha}^3, \ldots, \hat{\alpha}^n$.

Thus, since the trace of the commutator is identically zero, $Tr[\hat{\alpha}, (\Delta_2)^n \cdot \gamma_5] \equiv 0$, we get finally

$$\begin{aligned} Tr \exp(-t\,\check{\Delta}^2(\alpha)) &= Tr \sum_{n=0}^{\infty} \frac{1}{n!} (-t)^n (\check{\Delta}^2(\alpha))^n \\ &= Tr \exp(-t\,\Delta^2) + F(\hat{\alpha}^2, \ldots), \end{aligned} \quad (47)$$

where

$$F(\hat{\alpha}^2, \ldots) = Tr\left(\sum_{n=0}^{\infty} \frac{1}{n!} (-t)^n f^{(n)}\right).$$

Consequently, we have

$$\frac{\delta}{\delta\alpha(x)} Tr \exp(-t\,\check{\Delta}^2(\alpha))|_{\alpha=0} = 0,$$

and since the (divergent) determinant can be defined by a well known formula,

$$\ln \det \check{\Delta}^2(\alpha) = -\int_0^\infty \frac{dt}{t} Tr \exp(-t\,\check{\Delta}^2(\alpha)),$$

we get for (43) the final answer

$$\langle \nabla_\mu J^\mu_R \rangle = \frac{1}{2} \frac{\delta}{\delta(\alpha)} \ln \det \check{\Delta}^2(\alpha)|_{\alpha=0} = 0. \quad (48)$$

Hence, we have proved that the axial right current has no anomalies in the presence of the classical gravitational field. Note that the result (48) is valid in general, in any loop-approximation.

The absence of anomalies for the right γ_5-current is naturally explained, if one notice that the operator Δ (29) transforms γ_5-odd wave functions into γ_5-odd ones and analogously even to even ones.

5. Conclusion

We have studied in this paper the gravitational interaction of antisymmetric tensor fields which in the Ivanenko-Landau-Kähler approach describe the system of fermion particles. It turns out that contrary to the flat space theory, in which these fermions decouple and are reduced to the four independent Dirac spinor fields, in an arbitrary curved manifold the ATF-fermions interact in a non-trivial manner with the metric and connection. An important consequence of such interaction is the emergence of the effective non-linear self-coupling of the fermion fields. In our opinion the above obtained generalized nonlinear fermionic theory can be used in construction of sub-quark models (cf. e.g. ref. [26]). The interesting possibility of averting the cosmological singularity is now under consideration and preliminary results (to be published elsewhere) show that the ATF-fermions can prevent collapse in a class of spatially homogeneous cosmological models.

Of course, the above investigation is only a small part of the study of the properties of the spin-coupling between the fermions and the gravitational field. There still remain the problems of obtaining the concrete physical effects within the framework under consideration, the self-consistent quantisation of both the gravity and the fermion matter, the renormalizability problem etc. It seems also worth studying the supergravity models with ATF-fermions, thus describing not only fields, which mediate the interaction, but also the fundamental half-integer spin matter in purely geometrical terms.

In conclusion we want to point out on the fundamental question to be solved. In the flat space-time the ordinary quantum electrodynamics with the standard Dirac fields is indistinguishable from the QED with ATF-fermions. Moreover it is easy to obtain the non-abelian generalization, assuming that the ATF-fermions carry additional internal indices and transform under a representation of some internal (colour, flavour etc.) symmetry group, and hence one can construct alternative electroweak and grand unified models with ATF-fermions instead the Dirac spinors. Thus should we live in the Minkowsky space-time we could never get to know what kind of matter we and the whole Universe is constructed from: either Dirac or ATF fermions. However gravity, as we have seen above, feels the difference between these two types of half integer spin particles. Hence only gravitational experiments can give a definite answer on the important question: from which matter the Universe is built? This is one of the most impressive conclusions, since it is commonly assumed (especially by the elementary particle physicists) that the gravitation does not play an important role in the high energy physics due to the small value of the gravitational coupling constant $\varkappa$.

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