General Exact Solution to the Problem of the Probability Density for Sums of Random Variables

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The exact explicit expression for the probability density $p_N(x)$ for a sum of $N$ random, arbitrary correlated and arbitrary distributed random summands is obtained. The expression is valid for any number $N$ and any distribution of the random summands. Most attention is paid to application of the developed approach to the case of independent and identically distributed summands. The obtained results reproduce all known exact solutions valid for the, so called, stable distributions of the summands. It is also shown that if the distribution is not stable, the profile of $p_N(x)$ may be divided into three parts, namely a core (small $x$), a tail (large $x$), and a crossover from the core to the tail (moderate $x$). The quantitative description of all three parts as well as that for the entire profile is obtained. A number of particular examples are considered in detail.

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Here and in what follows $\langle \cdots \rangle$ means average over the corresponding random variable(s). Note that the assumption $\langle \xi \rangle = 0$ does not restrict generality of the consideration. If $\langle \xi \rangle \neq 0$, a new variable $\xi' \equiv \xi - \langle \xi \rangle$ may be introduced. Then, all the consideration should be conducted in terms of $\xi'$.

I am interested in $p(x)$ for the sum $x$. It is well known [1,2] that, generally speaking, $p(x)$ differs from $f(\xi)$. Namely, if the variance of the summands $\sigma^2 \equiv \langle \xi^2 \rangle$ is finite, then in the limit $N \to \infty$ $p_N(x)$ converges to GL

$$p_G(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right),$$

regardless of the particular type of $f(\xi)$. If the variance diverges (it means that at $\xi \to -\infty$, $\xi \to \infty$, or both $f(\xi)$ decays as $|\xi|^{-(\alpha+1)}$; $0 < \alpha \leq 2$) the corresponding limit PD is given by the Lévy distributions [1]. In the latter case the definition of the normalized variable $z$ is a bit tricky. I will return to this question later.

However, in any practical problem $N$ is a finite quantity. Then, the question about the actual PD for the sum at finite $N$ arises. In what follows I am going to derive the expression for the desired PD, which is valid for any $N$ and any $f(\xi)$. The expression depends on $N$ explicitly. To emphasize this dependence I will name the corresponding PD $p_N(z)$ while $p(z)$ will still denote the limit distribution at $N \to \infty$.

Let us first calculate $p_1(x)dx$—the probability that the sum $x$ has a certain large (compared, e.g., to the standard deviation) value from the interval $x_L \leq x \leq x_L + dx$ just due to contribution of a single summand, whose value is much greater than all other terms in the sum. Naturally, the sum may be large due to the contribution of any number of summands. Calculating $p_1(x)$ I take into account only a single possibility from a variety and hence obtain a lower bound for $p_N(x)$. The probability that, say, the first term in

\[ x = \sum_{n=1}^{N} \xi_n \quad \langle \xi \rangle = \int_{-\infty}^{\infty} \xi f(\xi) d\xi = 0. \]
Eq. (1) has the value $\xi_1 = x$ is $f(x)dx$. The probability that all other summands have the absolute values much smaller than $|x|$ is $\left[\int_{-\xi}^{\xi} f(\xi') d\xi'\right]^{N-1}$, where $|\xi| \ll |x|$. At last, bearing in mind that any of the $N$ summands may be large, and that for the problem in question all these events are mutually exclusive I obtain

$$p_{1N}(x)dx = N\left[\int_{-\xi}^{\xi} f(\xi') d\xi'\right]^{N-1} f(x) dx = N\left[1 - \int_{-\infty}^{-\xi} f(\xi') d\xi' - \int_{\xi}^{\infty} f(\xi') d\xi'\right]^{N-1} f(x) dx = Nf(x) dx,$$ (3)

provided $|x|$ is so large that inequality $(N - 1) \times \left[\int_{-\xi}^{\xi} f(\xi') d\xi' + \int_{\xi}^{\infty} f(\xi') d\xi'\right] \ll 1$ holds.

A remarkable conclusion following from Eq. (3) is that if $f(\xi)$ has a heavy (i.e., decaying as a certain power of $|\xi|$) tail, $p_{1N}(x)$ also must have a heavy tail, decaying not faster than that of $f(x)$, cf. [3–5]. Thus, one can see that there is a simple generic reason for heavy tails to come into being, than that of $f(x)$. A more general conclusion is that at any $f(\xi)$ decaying slower than $\exp(-c\xi^2)$ with any positive $c$ PD $p_{1N}(x)$ must have a tail, which decays slower than the Gaussian; cf. Eqs. (2) and (3).

To obtain the explicit expression for $p_N(x)$ I must first reproduce some known results of probability theory since I need them for further analysis. To begin with, I calculate the characteristic function $\langle \exp(i\omega z) \rangle$, where $z$ is given by Eq. (1) and $\omega$ is a real parameter. Here I consider random summands with a finite variance. The case when the variance diverges will be discussed later. At finite $\sigma$ it is convenient to introduce instead of $x$ the normalized quantity $Z$ in the same manner as that in Eq. (2). According to the definition of the characteristic function $[1,2]$

$$\langle \exp(i\omega z) \rangle = \int_{-\infty}^{\infty} \exp(i\omega z)p_N(z)dz,$$ (4)

where $p_N(z)$ stands for the desired PD. On the other hand, straightforward calculations for a sum of i.i.d. random summands yield the following:

$$\langle \exp(i\omega z) \rangle = \prod_{n=1}^{N} \exp\left[\frac{i\omega}{\sigma\sqrt{N}} \xi_n\right]$$

$$= \prod_{n=1}^{N} \exp\left[\frac{i\omega}{\sqrt{N}} \xi_n\right]$$

$$= \left[\exp\left[\frac{i\omega}{\sqrt{N}} \xi_n\right]\right]^N.$$ (5)

Finally, equalizing Eq. (4) to Eq. (5) and applying the inverse Fourier transform I obtain

$$p_N(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-i\omega z + N \ln N(\omega)]d\omega,$$ (6)

where the notation

$$\ln N(\omega) = -\frac{\omega^2}{2} + o(\omega^2).$$ (10)

Equations (6) and (7) provide (under the specified condition of finiteness of $\sigma$) the desired exact solution to the problem valid at any $N$ and any $f(\xi)$. The only modification of the above expressions in the case of diverging $\sigma$ is the corresponding change of the normalized factor in the definition of $z$. As it has been said I will return to the case later.

To generalize the obtained results to a sum of arbitrary correlated summands, note that the only difference arising in this case is that now $\langle \cdot \cdot \cdot \rangle$ in Eq. (5) means $\int \cdot \cdot \cdot f_N(\xi_1, \xi_2, \ldots, \xi_N)d\xi_1d\xi_2 \ldots d\xi_N$, where $f_N$ stands for the corresponding multidimensional probability density. It is convenient to represent $f_N$ in the form

$$f_N = \mu(\xi_1, \xi_2, \ldots, \xi_N) + \prod_{n=1}^{N} f(\xi_n),$$ (8)

where

$$f(\xi_n) = \int f_N(\xi_1, \ldots, \xi_N)d\xi_1 \cdots d\xi_{n-1} \times d\xi_{n+1} \cdots d\xi_N.$$ (9)

Here $p_{0n}(z)$ is given by the right-hand side of Eq. (6). If the symmetry does not hold, Eq. (9) becomes more cumbersome.

Let us return to the case of i.i.d. summands and discuss some general features of the obtained PD. To this end first I focus on the behavior of $g_N(\omega)$ at small $\omega/\sqrt{N}$. Expansion of the integrand in Eq. (7) in powers of $\omega/\sqrt{N}$ yields the following:

$$N \ln g_N(\omega) = -\frac{\omega^2}{2} + o(\omega^2).$$ (10)

Truncation of terms of the order $o(\omega^2)$ in Eq. (10) reduces the expression for $p_N(z)$ to the following integral:

$$p_N(z) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2} (\omega + iz)^2 - \frac{z^2}{2}\right]d\omega.$$ (11)

The integral is taken by integration in the plane of complex $\omega$ along the contour shown in Fig. 1. The integrand is an analytic function inside the contour, hence the integral over the entire contour is zero. The integrals along the vertical segments vanish exponentially at $\omega' \rightarrow \pm \infty$. Therefore the integral Eq. (11) equals the one along
FIG. 1. Contour of integration for Eq. (11).

the line $-\infty < \omega' < \infty; \omega'' = -z$. It eventually gives rise to GL; see Eq. (2).

However, my goal is to obtain the applicability conditions for Eq. (2) rather than to reproduce the well-known result of the central limit theorem [1,2]. The transformation of integral Eq. (11) into the one in the complex plane implies that expansion Eq. (10) is now performed in the complex plane too. The applicability condition for the expansion says that $|\omega|^2/N \ll 1$, where now $\omega$ is a complex quantity. This condition must hold at least for those $\omega$'s, which make the main contribution to the integral. For the integral along the line $-\infty < \omega' < \infty; \omega'' = -z$ the main contribution is made by $\omega' = O(1), \omega'' = -z$. If $z = O(1)$, it gives rise to inequality $N \gg 1$, which is trivial. At $|z| \gg 1$ the desired applicability condition reads $|z|^2 \ll N$.

In other words, I have obtained that at $|z| = O(\sqrt{N})$ the expression Eq. (2) for $p_N(z)$ becomes invalid and a crossover from GL to a new asymptotic behavior must occur. The only exception to the rule is the case, when instead of approximate expression Eq. (10) one has identity $N \ln g_N(\omega) = -\omega^2/2$, i.e., $g_N(\omega) = \exp[-\omega^2/(2N)]$. It is easy to show [see Eq. (7)] that the corresponding $f(\xi)$ in this case is also a Gaussian. The latter reflects the known fact that GL is a stable distribution [1]. The next important point to be made is that since the new asymptotics begins at $|z| = O(\sqrt{N})$, it exists at finite $N$ only, being shifted to infinity at $N \to \infty$. To understand the behavior of this asymptotics let us consider several concrete examples of $f(\xi)$.

(i) $f(\xi)$ with a heavy tail. In particular, I consider $f(\xi)$ of the following type:

$$f(\xi) = \frac{l \sin \pi \xi}{\pi \xi^{2l+1}} ,$$

where $l$ is any positive integer number. At $l = 1$ the above PD is reduced to the Cauchy distribution with diverging variance and expectation. In this case $z$ in Eqs. (4) and (6) should be defined as $z = x/N$. The calculations of $p_N(z)$ at $l = 1$ are trivial and yield for $p_N(z)$ the Cauchy distribution too, since as well as GL the Cauchy distribution also is a stable distribution [1].

At $l \geq 2$ PD $f(\xi)$ is not stable. The asymptotics of $p_N(z)$ at $z \gg \sqrt{N}$ may be obtained by extension of the integration in Eq. (6) to the plane of complex $\omega$. Details of the calculations in this case are rather cumbersome and will be presented elsewhere. The calculations yield the following result:

$$p_N(z) \equiv \frac{\ln \sin \frac{\pi z}{\sqrt{N}}}{\pi z^2 N^{2l-1}} ,$$

where $\sigma$ stands for $\sqrt{(\sin^2 \theta)/\sin^2 \xi}$ and $z = x/\sqrt{N}$. This asymptotics coincides exactly with that given by Eq. (3) and clearly has the same meaning [comparing Eqs. (3) and (13) one must remember that $p_N(x) = p_N(z)dz/dx$ and $z = x/(\sigma \sqrt{N})$]. Plots of $p_N(z)$ obtained by numerical integration of Eqs. (6), (7), and (12) at $l = 2$ for two values of $N$ ($10^{-2}$ and $10^{-4}$, respectively) are presented in Fig. 2. Crossovers from GL to heavy tails are seen clearly. In agreement with the above discussion the larger is $N$, the longer the corresponding curve follows GL. Thus, if $f(\xi)$ has a power tail, the tail of $p_N(z)$ follows the same power law [3–5]. However, the coincidence of the two asymptotics is not the generic property of any $f(\xi)$ decaying slower than $\exp(-\gamma \xi^2)$. To illustrate the latter issue let us consider the following case:

(ii) $f(\xi) = 1/(\pi \cosh \xi)$. In this case tedious evaluation of integral Eq. (6) in a complex plane yields the following asymptotic expression for the tail:

$$p_N(z) \equiv \frac{N^{N/2}}{(N-1)!} z^{N-1} \exp\left(-\frac{\pi z \sqrt{N}}{2}\right) .$$

PD given by Eq. (14) is much greater than that following from Eq. (3). The plot of $p_N(z)$ at $N = 25$ is shown in Fig. 3.

(iii) Finally, I consider an example of $f(\xi)$, resulting in a tail for $p_N(z)$, which is lighter (i.e., decaying faster) than a Gaussian. Let us take, e.g., the uniform distribution

![Graph of uniform distribution](image-url)
It is evident that \( p_N(z) \) must vanish identically at \( |z| \geq z_m = \sqrt{N/(2\sigma)} \), i.e., at \( |x| \geq N/2 \). Let us show that \( p_N(z) \) given by Eq. (6) does satisfy this condition. For the specified \( f(\xi) \) simple calculations yield

\[
g_N(\omega) = \frac{2\sigma\sqrt{N}}{\omega} \sin\left(\frac{\omega}{2\sigma\sqrt{N}}\right), \quad \sigma^2 = \frac{1}{12},
\]

so that

\[
p_N(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2\sigma\sqrt{N}}{\omega} \sin\left(\frac{\omega}{2\sigma\sqrt{N}}\right)\right)^N e^{-i\omega z} \, d\omega.
\]

Let us extend the integration in Eq. (16) to the plane of complex \( \omega \) and note that the integrand is an analytic function of \( \omega \) at any finite \( |\omega| \). For this reason the corresponding integral over any closed contour in the complex plane is identical zero. For definiteness in what follows I suppose that \( z > 0 \) (the case of negative \( z \) is analyzed analogously). Then, I consider a closed contour consisting of two elements—a segment of the real axis \(-R \leq \omega \leq R\) and a circular arc with the radius \( R \) lying in the lower half-plane, which connects the edges of the segment. Next, note that the modulus of the integrand in the lower half-plane is not grater than

\[
\text{const} \cdot \frac{1}{|\omega|^N} \exp\left[-\omega'' \left(\frac{\sqrt{N}}{2\sigma} - z\right)\right], \quad \omega'' \leq 0.
\]

It is straightforwardly seen from the above expression that the integral over the arc tends to zero at \( z > \sqrt{N/(2\sigma)} \) and \( R \to \infty \). At last, bearing in mind that the integral over the entire closed contour is zero, I conclude that the integral Eq. (16) does transforms into identical zero at \( z > \sqrt{N/(2\sigma)} \). Next, applying the saddle-point method one can obtain [8] that when \( |z| \) approaches the marginal value \( z_m \) from below, \( p_N(z) \) decays as \( (1 - |z|/z_m)^{N-1} \). The behavior of \( p_N(z) \) as a whole at \( N = 10 \) is shown in Fig. 4.

Summarizing the obtained results, I can say that derived exact expressions Eqs. (6)–(9) do provide the quantitative description for the entire profile of \( p_N(z) \), valid for any number and any distribution of random summands.

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