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WHEELER–DE WITT SUPERSPACE AND TOPOLOGICAL TRANSITIONS IN GRAVITY THEORY

V. D. Dzhunushaliev and G. A. Sardanashvili

UDC 530.12

Despite all the success of the Einstein theory of gravitation, it contains a number of essential difficulties, and one of them is the problem of gravitational singularities. One of the proposed solutions has been to consider these singularities as a kind of topological transitions (TT), which we have to learn how to describe in the theory of gravity.

For the regular case, one can use the method of (3 + 1)-splitting and represent evolution in a theory of gravity as a certain trajectory in the Wheeler–De Witt superspace of 3-geometries, which at the same time is the space of states in the canonical quantization approach [1]. It seems natural to use a similar construction for the case of TTs as well, especially since a number of authors consider TTs in the extreme case of gravitational singularities as an essentially quantum process.

Without speaking so far about evolutionary equations which might describe such a process, let us consider the necessary condition for the very possibility of such a description: namely, the existence of a connected superspace which includes the spaces of 3-geometries both before and after the TT. Only if such a connected superspace exists, one can hope to connect the points of the 3-geometries before and after the TT by a continuous curve. Otherwise, most probably, one will have to be satisfied with defining a formal matrix of the TT probabilities, as is done, e.g., in [2].

Using the method of (3 + 1)-splitting, a regular spacetime can be represented as a spacetime foliation, whose leaves are three-dimensional spacelike hypersurfaces considered as surfaces of simultaneity. For any finite time interval, all the leaves of such a foliation are diffeomorphic to each other.

The presence of a gravitational singularity (if there is no causality violation) can be represented as the existence of a singularity of the spacetime foliation and the TT as a change in the topology of leaves of the foliation [3]. We shall assume that the TT takes place in a relatively compact region of spacetime and that the leaves of the foliation are compact, since if they are noncompact and if the metric at the infinity is isotropic, they can be compactified by adding the infinity point. Then the TT satisfies the following theorems [4]:

Definition. Two smooth compact oriented n-dimensional manifolds M_1 and M_2 are called orientedly cobordant if there exists a smooth compact oriented manifold N such that its boundary ∂N is orientedly diffeomorphic to the manifold $M_1 + (-M_2)$, where $(-M_2)$ means the manifold M_2 with the reversed orientation.

THEOREM. Any two three-dimensional compact manifolds are cobordant.

THEOREM. On each cobordism there exists a correct Morse function.

This means that there are no topological obstacles to changing the topology of spacelike leaves, and such a change can be represented as a chain of transitions through critical points of some Morse function on the spacetime manifold, the level surfaces of this Morse function being the leaves of the manifold. Therefore, it suffices to consider a TT through one critical point of the Morse function.

M. V. Lomonosov Moscow State University. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No. 12, pp. 73-75, December, 1986. Original article submitted October 22, 1984.

 0038-5697/86/2912-1014$12.50 © 1987 Plenum Publishing Corporation

By a choice of coordinates, the Morse function in the neighborhood of a critical point can be represented in the canonical form

$$f(x) = -\sum_{i=1}^{\kappa} x_i^2 + \sum_{j=\kappa+1}^{4} x_j^2.$$

The index k is uniquely determined by the critical point and completely determines the type of the TT through this point, according to the following theorem [5]:

THEOREM. Let us denote the leaf of the singular foliation under consideration before the transition by M_-, and the leaf after the transition through the critical point of index k, by M_+. Then there are cells e^k and e^{4-k} whose intersection $e^k \cap e^{4-k}$ contains only the critical point itself, and satisfying the conditions that $M_- \cap e^k = \partial e^k$, $M_+ \cap e^{1-k} = \partial e^{1-k}$, and that $M_- - \partial e^k$ is diffeomorphic to $M_+ - \partial e^{-k}$.

This means that the TT can be represented as a mapping of M_- into M_+ such that $M_- - \partial e^k$ is uniquely mapped onto $M_+ - \partial e^{4-k}$; $\partial e^k \subset M$ contracts into the critical point, and then expands to $\partial e^{4-k} \subset M_+$.

Consider now the behavior of Wheeler—De Witt superspaces under such a TT. Since the leaves of the spacetime foliation are diffeomorphic to some typical leaf M for the case of regular evolution, it can be described as a change of the space metric on M, or as a motion in the superspace S(M).

To define the superspace S(M), we shall first introduce a linear space $S_L(M)$ of symmetric covariant tensor field of rank 2 on M, and define the topology of uniform convergence on it in derivatives of arbitrary order. This topology induces a topology on the subspace R(M) of Riemannian metrics on M, which forms an open cone in $S_L(M)$. The factorspace R(M) over the group of diffeomorphisms of the manifold M is called the Wheeler—De Witt superspace S(M). Generally, however, the superspace S(M) has a bad structure, and it is often more convenient to deal with the space R(M), which is a Frechet manifold, metrizable and connected [6].

In the case of a TT, when there are two nondiffeomorphic typical leaves M_- and M_+, we have the problem of gluing together the superspaces $S(M_-)$ and $S(M_+)$ or spaces $R(M_-)$ and $R(M_+)$. The gluing can be accomplished using the diffeomorphism of the spaces $M_- - \partial e^k$ and $M_+ - \partial e^{4-k}$. To do it, let us consider in the space $S_L(M_-)$ the subspace T_- of tensor fields which are Riemannian metrics on $M_- - \partial e^k$, and whose components all vanish on ∂e^k. The points of T_- are boundary points with respect to the open cone $R(M_-)$ in the space $S_L(M_-)$, but the space T_- itself is not closed in $S_L(M_-)$.

Let us construct a similar space $T_+ \subset S_L(M_+)$ of Riemannian metrics $M_+ - \partial e^{4-k}$, vanishing on ∂e^{4-k}. The spaces T_+ and T_- are homeomorphic to each other. Let us glue the spaces $R(M_-) \cup T_-$ and $R(M_+) \cup T_+$ over the boundary points of T_- and T_+, identifying these points and defining the topology of gluing on the resultant space R. In this topology, the neighborhoods of a point g obtained by identifying the points G_- and G_+ from T_- and T_+, are all sets whose intersections with the spaces $R(M_-) \cup T_-$ and $R(M_+) \cup T_+$ imbedded in R, are the neighborhoods of g_- and g_+ in $R(M_-) \cup T_-$ and $R(M_+) \cup T_+$, respectively.

The space R is connected (though it need not be a manifold at the glued points), and the change of the metric during the TT can be described by a trajectory in R connecting the points of $R(M_-)$ and $R(M_+)$.

Let us now construct a superspace of TT of index k. To do that, let us consider the group G_- of diffeomorphisms of the leaf M_-, leaving the boundary $\partial e^k \subset M_-$ invariant, and the group G_+ of diffeomorphisms of the leaf M_+, leaving the boundary $\partial e^{4-k} \subset M_+$ invariant. The actions of the group G_- on the space T_- and of the group G_+ on the space T_+ are equivalent. Therefore, one can define the factorspace of the space R with respect to the action of the groups G_- and G_+ in R, which can be considered as a generalization of the Wheeler—De Witt superspace to the case of TT. The points from $T_\pm/G_\pm$ are not isolated in S, and there are trajectories in S connecting the points of 3-geometries before and after the TT.

Let us emphasize that the superspace S is not obtained by gluing the superspaces $S(M_-)$ and $S(M_+)$ together. The reason is that we considered a TT at a fixed spacetime point with contraction and expansion of the leaves along fixed cells e^k and e^{4-k}. Therefore, the factorspaces of the spaces $R(M_-)$ and $R(M_+)$ were not taken with respect to the groups of all diffeomorphisms of the leaves M_- and M_+, but rather with respect to smaller groups G_- and G_+. If necessary, one can also consider the case of a "wandering" TT point.

Thus, we showed that it is possible in principle to describe the evolution in the theory of gravity through TTs. The next step of the description of TTs should be a construction of related evolutionary equations.

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EXACT SOLUTIONS OF THE EQUATIONS OF MOTIONS OF QUARKS IN A NON-ABELIAN FIELD OF A FLAT COLOR WAVE

I. A. Obukhov, V. K. Perez-Fernandez, and V. R. Khalilov

UDC 539.1.01

Exact solutions of the equations of motion of quarks in a non-Abelian field of a flat color wave of a spectral configuration are obtained. The gauge field of the wave takes values from the SU(2) group and is an exact solution of the Yang–Mills equations.

As shown in [1], the flat-wave solution of the Yang–Mills equations can be written as:

$$A_\mu^a = [f^a(\xi)(n^1\cdot x) + g^a(\xi)(n^2\cdot x) + h^a(\xi)]\,\kappa_\mu. \tag{1}$$

Here A_μ^a are the vector potentials of the Yang–Mills fields; a is the color index; n_μ^1 and n_μ^2 are the spacelike 4-vectors specifying the polarization of the color wave; κ_μ is the isotropic wave 4-vector; $\xi = (\kappa\cdot x)$ is the wave phase; $(a\cdot b) = g_{\mu\nu}a^\mu b^\nu$ is the scalar product in the Minkowski space with diag $g_{\mu\nu} = (1, -1, -1, -1)$; $f^a(\xi)$, $g^a(\xi)$, and $h^a(\xi)$ are arbitrary bounded functions of the phase ξ; $(n^1\cdot n^2) = (\kappa\cdot n^1) = (\kappa\cdot n^2) = 0$.

If the energy density of the Yang–Mills fields is bounded, the direction of the Poynting vector is constant, and its norm numerically equals the energy density, then the solution (1) is unique modulo $h^a(\xi)$ [1].

The functions $h^a(\xi)$ in expression (1) are a purely gauge artifact and can be eliminated by a gauge transformation which depends only on ξ. The existence of such a transformation follows from the gauge freedom remaining after applying to the vector potentials A_μ^a the Lorentz condition:

$$\partial_\mu A^{a\mu} = 0.$$

M. V. Lomonosov Moscow State University. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No. 12, pp. 76-78, December, 1986. Original article submitted October 31, 1984.

0038-5697/86/2912-1016$12.50

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