ISSN 0027-1349, Moscow University Physics Bulletin, 2014, Vol. 69, No. 5, pp. 421–425. © Allerton Press, Inc., 2014.
Original Russian Text © V.P. Yushkov, 2014, published in Vestnik Moskovskogo Universiteta. Fizika, 2014, No. 5, pp. 55–59.

PHYSICS OF THE EARTH, ATMOSPHERE, AND HYDROSPHERE

Towards the Proof of Kolmogorov Hypotheses

V. P. Yushkov

Department of Physics, Moscow State University, Moscow, 119991 Russia

e-mail: yushkov@phys.msu.ru

Received February 12, 2014; in final form, June 6, 2014

Abstract—A closure scheme for Kolmogorov spectrum at low and high frequencies is proposed. It allows us to validate second Kolmogorov hypothesis if expand the first one. The proposed closure scheme adds energy of turbulence to the list of controlling parameters and explains energy transfer over the spectrum by wave interaction between incompressible and adiabatic components of turbulence.

Keywords: turbulence, spectrum, dissipation, approximation.

DOI: 10.3103/S0027134914050105

INTRODUCTION

In 1941, Kolmogorov proposed two hypotheses [1], which formed the basis of turbulence theory [2, 3]. However, throughout the years of intensive studies, these hypotheses have never been rigorously proven [4], although they were experimentally confirmed in the "−5/3 law." In this paper a way of proving of second Kolmogorov hypothesis is outlined if we expand the first one. This complement makes it possible to describe the entire spectrum (in both the viscous and the energy ranges), as well as to use inverse Fourier transform to explain observed deviation of structural and correlation functions from Kolmogorov law [5].

In the extended formulation, the first similarity hypothesis is written as follows: "For the locally isotropic turbulence the distribution of F_n is uniquely determined by the quantities ν, ε, and E." We have added the turbulence energy, E to Kolmogorov's formulation. The key role of turbulence energy in the small-scale fluctuations spectrum formation was always beyond doubt, but the addition of this parameter destroys the assumption on the unicity of the spectrum shape, which is derived from the dimensional analysis. In the dimensionless coordinates, the spectrum can also contain a function of the dimensionless ratio $E/\sqrt{\varepsilon\nu}$, where ε is the dissipation rate and ν is the kinematic viscosity. This ratio in turn depends on the Reynolds number.

To validate Kolmogorov's second hypothesis, we proceed from the universality of the kinetic energy dissipation spectrum $\varepsilon(\kappa) = \kappa^2 E(\kappa) = 4\pi\kappa^4\Phi(\kappa)$, where $E(\kappa)$ is the kinetic energy spectral density of uniform and isotropic stationary turbulence. The Fourier transform of the correlation function is expressed trough $\Phi(\kappa)$ [2]. The universality of dissipation rate spectrum $\varepsilon(\kappa)$ is certainly a hypothesis, but this hypothesis is generally adopted from the physical assumption that, at small scales the mechanism of the mixing and energy transfer over the spectrum is the same and independent of external conditions. However, the spectrum width depends on the Reynolds number, which relates the internal and external (integral) scales of turbulence.

In a spatial spectral representation [6], the kinetic energy of turbulence is expressed through $E(\kappa)$ as follows:

$$E = \int_0^\infty E(\kappa)d\kappa. \tag{1}$$

However, kinetic energy of Lagrange particles is not rigorously conserved, since it can transform to potential energy, which, in turn, can transform to heat (internal) energy. We will not consider a relationship between the kinetic and total energies of turbulence. We only assume that these energies are statistically proportional to each other. This assumption allows us to remain within Kolmogorov's representation and not to go to frequency distributions of energy fluctuations. At the same time, the closure we propose in this work can be only proved by using the connection between kinetic and internal or *available potential* energies of turbulence, as well as of a finitness of turbulent irregularities lifetime.

The closure of the turbulence spectrum on small and large scales rests on the assumptions formulated in [7], and on the energy transfer over the spectrum via the wave interaction of incompressible turbulent movements (Kolmogorov turbulence) with fast adiabatic oscillations, i.e., acoustic and acoustic–gravity waves [7]. Fast (high-frequency) and compressible waves transfer the energy and momentum and smooth the physical properties of incompressible Lagrangian particles. The mechanism of this mixing is well known.

It was proposed by Tatarskii [8] and it is the Bragg's or resonance interaction of acoustic waves with turbulence. Unlike the traditional Obukhov's model [6], as well as the models that were proposed by other authors (for further details see [2]), the energy flow over the spectrum is related to probabilistic transition between two arbitrary wavenumbers κ_i and κ_j rather than to energy transfer through the wavenumber κ. The statistical parametrization of these resonance transitions, which can be represented as the interaction of the "turbulence diffraction gratings" with fast adiabatic movements, leads to the entropy dissipation of the Lagrangian particles. The dissipation rate statistically characterizes the incompressible turbulent fluctuation lifetime [9]. Nonlinear wave interaction can also be described via the Hamiltonian formalism of the interaction of incompressible turbulence with acoustic waves [10].

THE SPECTRAL REPRESENTATION AND INTEGRAL RELATIONS

In uniform and isotropic turbulence the kinetic energy is expressed through the correlation function of the Lagrangian particles velocities B_{ij} at zero: $E = B_{ii}(0)$, where $B_{ij}(\Delta \mathbf{r}) = \langle u_i(\mathbf{r})u_j(\mathbf{r}')\rangle$ and $\Delta \mathbf{r} = |\mathbf{r}' - \mathbf{r}|$. The angular brackets mean the statistical averaging and subscripts denote the orthogonal components of velocity vector, $\mathbf{u}$, repeated indices are implicitly summed over (i.e. E is the trace of the correlation tensor).

The convergence of integral (1) does not result from Kolmogorov's hypotheses since Kolmogorov considers the structural function of the velocities [2] $W_{ij}(\Delta \mathbf{r}) = 2(B_{ij}(0) - B_{ij}(\Delta \mathbf{r}))$ rather than the correlation functions. In Kolmogorov theory, the energy of turbulence can take an infinitely large value. Note that the correlation and structural functions have the same dimension; therefore, the behavior of the velocity structure function in the inertial range based only on dimension analysis cannot explain the behavior of the correlation function in this range. It suggests that $B_{ii}(0)$ or the energy of turbulent mixing is the governing characteristic of the turbulent fluctuations spectrum.

In Kolmogorov model, the velocity spectral density $E(\kappa)$ determines one more important integral, namely, the dissipation rate:

$$\varepsilon = 2\nu \int_0^{\infty} \kappa^2 E(\kappa) d\kappa. \quad (2)$$

A key feature of integrals (1) and (2) is the following obvious corollary of the Kolmogorov–Obukhov law: the "–5/3 law" makes it impossible to determine the energy of turbulence and the dissipation rate by Kolmogorov spectral density integration [8]. If $E(\kappa) \sim \kappa^{-5/3}$, integral (1) diverges at zero and integral (2) diverges at $\kappa \longrightarrow \infty$. To overcome this obvious obstacle, not long after the publication of Kolmogorov's and Obukhov's results, several model closure schemes of a "universal" turbulence spectrum in the viscous and energy ranges were proposed [2]. These schemes have not completely been validated physically as yet, but they are essentially necessary for calculating of waves propagation in a turbulent medium; therefore, they are used in practice as "empirical" ones [11].

To combine observed behavior of the velocity spectrum and the effect of adiabatic fluctuations in the high-frequency range on this spectrum, the following closure formula is proposed:

$$E(\kappa) = C\varepsilon^{2/3}(\kappa_0 + \kappa)^{-5/3}\left[\frac{\delta\kappa}{e^{\delta\kappa} - 1}\right]. \quad (3)$$

If $\frac{1}{\delta} \gg \kappa \gg \kappa_0$, $E(\kappa) \sim \kappa^{-5/3}$ and it describes the empirical Kolmogorov–Obukhov law. At $\delta \longrightarrow 0$, the term in the braces tends to unity and can be considered as a correction factor in Kolmogorov theory. This term is necessary, since the parameter ε in (3) is determined by the behavior of the spectrum at high frequencies; at $\kappa \longrightarrow \infty$ the spectrum is governed by the damping of fast adiabatic oscillations, which transfer and smooth the energy and momentum fluctuations of the Lagrangian particles [7].

The parameter $\kappa_0 = 2\pi/L_0$, where L_0 is the integral scale of turbulence, is required to limit the Kolmogorov spectrum at zero. Unlike von Karman spectrum [12] we limit the scalar spectral density $E(\kappa)$ rather than $\Phi(\kappa)$. The latter function always enters into three-dimensional spectral integrals with the multiplier κ^2. The finiteness of the small-scale turbulence spectral density $E(\kappa)$ at zero are confirmed by numerous observations, both classical [13, 14] and current [15]. The value of $E(0)$ is an important characteristic of the spectrum since it determines the convergence of the correlation function $B_{ii}(r)$.

In the uniform and isotropic turbulence E and B are related by the following relation:

$$B_{ii}(r) = \int_{-\infty}^{+\infty} \cos(\kappa r) E(\kappa) d\kappa. \quad (4)$$

The Fourier transform makes it possible to calculate the correlation and spectral functions using spectrum (3) and shows that the structure function behavior in the vicinity of zero is determined by the spectrum behavior in the vicinity of κ_0. Formula (3) can also be considered just as a convenient approximation. Other variants of the spectrum in the low-frequency range (at

$0 < \kappa < \kappa_0$), with Obukhov's limit $E(\kappa) \sim \kappa^{-5/3}$ at $\kappa \gg \kappa_0$, will only give a correction factor of the order of unity in calculations (1) and (2).

Two parameters of spectral distribution (3) (κ_0 and δ) are uniquely related to integral characteristics of the spectrum (E and ε). Under increasing Reynolds number and the dimensionless quantity L_0/δ, the dependence of δ on E is decreased and δ becomes a function of ε and ν only. In the inertial range, if $\kappa_0 \ll \kappa \ll 1/\delta$ and $L_0/\delta \longrightarrow \infty$, the dependence of the spectral density on both ν and E falls out of (3), which proves Kolmogorov's second hypothesis.

CONVERGENCE TO THE KINETIC ENERGY OF TURBULENCE

Let us consider the integral

$$E = C\varepsilon^{2/3}\int_0^{\infty}\frac{(\kappa_0+\kappa)^{-5/3}\delta\kappa}{e^{\delta\kappa}-1}d\kappa. \quad (5)$$

The substitution of variables $\kappa = \kappa_0 t$ yields the following expression:

$$E = C\varepsilon^{2/3}\kappa_0^{-5/3}\kappa_0^2\delta\int_0^{\infty}\frac{(t+1)^{-5/3}t}{e^{\delta\kappa_0 t}-1}dt. \quad (6)$$

We also introduce the following designation of the dimensionless number: $\kappa_0\delta \equiv \Delta$. As the Reynolds number increases, $\Delta \longrightarrow 0$. Let us now consider the following integral I:

$$I = \Delta^{1/3}\int_0^{\infty}\frac{(t+1)^{-5/3}t}{e^{\Delta t}-1}dt = \Delta^{-2/3}\int_0^{\infty}(t+1)^{-5/3}\left[\frac{\Delta t}{e^{\Delta t}-1}\right]dt. \quad (7)$$

We divide this integral into two parts (I_1 and I_2) with different behaviors at $t \longrightarrow \infty$ and $\Delta \longrightarrow 0$:

$$I_1 = \Delta^{-2/3}\int_0^{m}(t+1)^{-5/3}\left[\frac{\Delta t}{e^{\Delta t}-1}\right]dt, \quad (8)$$

and

$$I_2 = \Delta^{-2/3}\int_m^{\infty}(t+1)^{-5/3}\left[\frac{\Delta t}{e^{\Delta t}-1}\right]dt. \quad (9)$$

The expansion of the function in the braces into the Taylor series at $t < m$ and $\Delta \longrightarrow 0$ so that $\Delta m \ll 1$ yields the following expression:

$$\begin{aligned} I_1 &= \Delta^{-2/3}\int_0^{m}(t+1)^{-5/3}[1-\Delta t/2+o(\Delta t)]dt \\ &\approx \Delta^{-2/3}\left[\int_0^{m}(t+1)^{-5/3}dt - \frac{\Delta}{2}\int_0^{m}(t+1)^{-5/3}t\,dt\right] \\ &= \Delta^{-2/3}\left[\int_0^{m}(t+1)^{-5/3}dt - \frac{\Delta}{2}\left(3(m+1)^{1/3}+\frac{9}{2}+\frac{3}{2}(m+1)^{-2/3}\right)\right]. \end{aligned} \quad (10)$$

It can be seen that at $\Delta \longrightarrow 0$ the remainders decrease, except for the first one.

Let us make an upper estimate of the second integral. Since $\left|\frac{\Delta t}{e^{\Delta t}-1}-1\right| \le 1$, then

$$\begin{aligned} I_2 &= \Delta^{-2/3}\left[\int_m^{\infty}(t+1)^{-5/3}dt + \int_m^{\infty}(t+1)^{-5/3}\left[\frac{\Delta t}{e^{\Delta t}-1}-1\right]dt\right] \\ &\approx \Delta^{-2/3}\left[\int_m^{\infty}(t+1)^{-5/3}dt + \frac{3}{2}(m+1)^{-2/3} + O((m+1)^{-2/3})\right]. \end{aligned} \quad (11)$$

The first term in braces is retained to express the general integral through

$$\int_0^{\infty}(t+1)^{-5/3}dt = 3/2.$$

The remainders are of the order of $(m+1)^{-2/3}$; at $m \longrightarrow \infty$, these remainders are much less than unity. If the remainders in the integral I_1 are also to be less than the principal term and to decrease with the same rate as the reminders in I_2 does, it is sufficient to set $\Delta(m+1) = \text{const} \ll 1$.

Thus, at $\Delta \longrightarrow 0$, only the first term can be retained in the integral I and the other terms can be dropped. In this case the integral I is expressed as follows:

$$I \approx \Delta^{-2/3}\int_0^{\infty}(t+1)^{-5/3}dt = \frac{3}{2}\Delta^{-2/3}. \quad (12)$$

Then we obtain:

$$E = \frac{3}{2}C\varepsilon^{2/3}\delta^{2/3}\Delta^{-2/3} = \frac{3}{2}C\left(\frac{\varepsilon}{\kappa_0}\right)^{2/3}, \tag{13}$$

or, within numerical factor of the order of unity the integral scale of turbulence is $L_0 = 2\pi\frac{E^{3/2}}{\varepsilon}$. It is obvious that L_0 is not implicitly dependent on ν at $\Delta \longrightarrow 0$.

CONVERGENCE TO THE DISSIPATION RATE

Let us now consider integral (2). When the spectrum $E(\kappa) \sim \kappa^{-5/3}$ is used, this integral diverges at $\kappa \longrightarrow \infty$. The Gaussian truncation at the high-frequency band of the spectrum, which was proposed by Novikov and used by Tatarski, is not validated from the physical viewpoint and does not correspond to the observations [2, 12]. This formula also requires a special truncation scale to be introduced. Approximation (3) proposed in this work does not require this scale and explains the limitation of the dissipation rate spectrum $\varepsilon(\kappa)$ at $\kappa \longrightarrow \infty$ by the damping of high-frequency adiabatic waves. In the "viscosity range" the dissipation rate decreases nearly exponentially, as the spectrum of adiabatic oscillations, which is concentrated in the high-frequency range and has the Planck form [7]. The assumption of the Planck curve for the adiabatic turbulent fluctuations spectrum was made by Obukhov in 1941 [6]. In other words, the proposed hypothesis of closure has a physical explanation rather than being only the corollary of mathematical convergence requirements or "empirical" functions.

Using the above-introduced notations, we obtain the following expression:

$$\varepsilon = 2\nu C\varepsilon^{2/3}\delta^{-4/3}J, \tag{14}$$

where

$$J = \int_0^\infty \frac{t^2}{(\Delta+t)^{5/3}}\frac{t}{e^t-1}dt, \tag{15}$$

but here, $t = \delta\kappa$.

We also divide this integral into two parts (J_1 at $t < m < 1$) and (J_2 at $t > m$), which demonstrate different behaviors at $\Delta \longrightarrow 0$ and $t \longrightarrow \infty$. In the vicinity of zero, we have $\frac{t}{e^t-1} \lesssim 1$. Therefore,

$$J_1 \lesssim \int_0^m t^2(t+1)^{-5/3}dt, \tag{16}$$

if $m < 1$. Introducing $x = \Delta + t$ and shifting the limits, we obtain the following estimation:

$$\begin{aligned} J_1 &\lesssim \int_\Delta^{\Delta+m}\frac{(x-\Delta)^2}{x^{5/3}}dx = \int_\Delta^{\Delta+m}(x^{1/3} + 2\Delta x^{-2/3} \\ &+ \Delta^2x^{-5/3})dx = \left(\frac{3}{4}x^{4/3} - 6\Delta x^{1/3} - \frac{3}{2}\Delta^2x^{-2/3}\right)\Bigg|_\Delta^{m+\Delta} \\ &= O(m^{4/3}), \end{aligned} \tag{17}$$

if $\Delta < m$.

The principal value of the integral J is determined by the value of J_2, i.e., by the behavior of the integrand at $t \longrightarrow \infty$:

$$J_2 = \int_m^\infty \frac{t^{4/3}}{e^t-1}\left[\left(\frac{t}{\Delta+t}\right)^{5/3}\right]dt. \tag{18}$$

If $t > m$ and $\Delta < m$, the expression in the braces can be estimated as $1 - \frac{5}{3}\frac{\Delta}{m} + o\left(\frac{\Delta}{m}\right)$. Let us introduce the additional designation

$$J_0 = \int_0^\infty \frac{t^{4/3}}{e^t-1}dt = \Gamma(7/3)\zeta(7/3) \approx 1.658,$$

where $\Gamma(x)$ is the gamma function and $\zeta(x)$ is the Riemann zeta function. Then we obtain the following expression:

$$\begin{aligned} J_2 &\approx \left[1 - \frac{5}{3}\frac{\Delta}{m}\right]\int_m^\infty \frac{t^{4/3}}{e^t-1}dt \\ &= \left[1 - \frac{5}{3}\frac{\Delta}{m}\right]\left(J_0 - \int_0^m \frac{t^{4/3}}{e^t-1}dt\right). \end{aligned} \tag{19}$$

The last correction term for J_0 can easily be estimated by expanding the exponent into a Taylor series. It is approximately equal to $0.75m^{4/3}$ at $m \ll 1$. In other words, at $\Delta \longrightarrow 0$, m should be selected so that $\Delta \ll m \ll 1$. Then $J_1 \ll J_2$ and $J \approx J_0$. For example, a power dependence between m and Δ can be established as $m \sim \Delta^{3/7}$, so that at $\Delta \longrightarrow 0$ the remainders in (16) and (18) decrease with the same rate.

For now, the relation between ε, ν, and δ can be found. From (14), $\varepsilon^{1/3} = 2C\nu J_0\delta^{-4/3}$, and we obtain the following relation:

$$\delta = \sqrt[4]{(2CJ_0\nu)^3/\varepsilon}. \tag{20}$$

This formula shows that the parameter δ of spectral density (3) is proportional to Kolmogorov's internal scale. In this case, there is no need for introducing an additional scale for the "viscosity range." The empirical coefficient C was introduced to adjust the spectrum

(3) to Kolmogorov one, because the damping of fast adiabatic oscillations is caused not only by the kinematic viscosity ν, but also by the thermal diffusivity χ. For adiabatic movements, the effective loss factor is $\eta = \frac{4}{3}\nu + \left(\frac{C_p}{C_v} - 1\right)\chi$, where C_p/C_v is the ratio of air heat capacities at constant pressure and volume (density) [16].

CONCLUSIONS

Model (3) of the small-scale turbulence spectrum that was presented in this paper is a statistical model. This means that the integral characteristics ε and E, which determine the parameters of the spectrum κ_0 or L_0 and δ remain unchanged within the statistical ensemble of turbulent fluctuations. The phenomenon of intermittency, e.g., fast and random change in parameter ε, observed in atmospheric turbulent flows cannot be explained in the context of Kolmogorov approach.

To explain the phenomenon of intermittency, it is necessary to go beyond the limits of the incompressibility and stationary approximation and to consider the relation between the kinetic and total energies of turbulence. The total energy of turbulence also includes the thermal and potential components. These components can be joined in the concept of the available potential energy of turbulence [17], which is the fraction of the total energy that can freely transform to kinetic energy and back, resulting in intermittency. Similarly, the dissipation of the kinetic energy ε is not only component of the total dissipation of turbulent fluctuations. One would expect that the turbulent kinetic energy that is experimentally determined using formula (13) is a fluctuating characteristic even if the total energy of turbulence is conserved. Experimental observations show that these fluctuations are appeared in calculations of structure function of temperature or sound velocity fluctuations [9] and in the refraction index of turbulent atmosphere [11]. Strong fluctuations of the temperature structure function exponent in the atmospheric boundary layer were observed by Tatarski as early as 1956 [2].

The spectrum parameters κ_0 and δ depend on statistical properties of turbulence and should be determined experimentally or they can be parametrized in the models depending on the thermal and velocity stratification. For example, parameter κ_0 can be estimated by so-called compensated ($fE(f)$) frequency spectra of small-scale turbulent fluctuations [14] using Taylor's hypothesis: $f = U\kappa$, where U is the average velocity of the turbulent flow. If f_m is the maximum of such a spectrum and $f_m = U\kappa_m$, it can easily be calculated that $\kappa_m = 1.5\kappa_0$ if $\delta\kappa_0 \ll 1$.

The parameter δ (internal scale) in spectrum (3) characterizes the behavior of the spectrum of adiabatic fluctuations [7]. In the Plank approximation of this spectrum, $\delta \approx \gamma c_s/C_v T$, where γ is the damping parameter of the entropy fluctuations of the Lagrange particles and c_s is the average sound velocity.

This work does not present a rigorous proof of the second Kolmogorov hypothesis. It just outlines a way towards such proof. For example, the question is topical of why the exponent is 5/3 in the observations and in spectrum (3) if the dimensional analysis offer other possible combinations. Possibly, the answer can be found using the concept of redistribution of total energy over three components (kinetic, potential and internal) in view of the anisotropy of turbulent mixing.

REFERENCES

1. A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941).
2. A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics. Vol2: Mechanics of Turbulence* (MIT Press, Cambridge, Mass., 1971).
3. U. Frish, *Turbulence* (Cambridge, 1995).
4. A. Yaglom, *The Century of Turbulence Theory: The Main Achievements and Unsolved Problems* (EDP Sciences. Springer-Verlag, 2001).
5. L. Tan, W. Du, J. Ma, S. Yu, and Q. Han, Optics Express **18**, 451 (2010).
6. A. M. Obukhov, Izv. Akad. Nauk SSSR, Ser. Geogr. Geofiz. **5**, 453 (1941).
7. V. P. Yushkov, Mos. Univ. Phys. Bull. **68**, 330 (2013).
8. V. I. Tatarski, *Effects of the Turbulent Atmosphere on Wave Propagation* (Keter Press, Jerusalem, 1971), *Rasprostranenie voln v turbulentnoi atmosfere* (Propagation of Waves in Turbulent Atmosphere) (Moscow, 1967) [in Russian].
9. V. P. Yushkov, Mos. Univ. Phys. Bull. **67**, 384 (2012).
10. V. E. Zakharov, V. S. L'vov, and G. E. Falkovich, *Kolmogorov Spectra of Turbulence. I. Wave Turbulence* (Springer Series in Nonlinear Dynamics, Springer-Verlag, 1992).
11. E. Goldbraikh and N. S. Kopeika, Appl. Opt. **453**, 6151 (2004).
12. D. K. Wilson, J. G. Brasseur, and K. E. Gilbert, J. Acoust. Soc. Am. **10**, 30 (1999).
13. J. C. Kaimal, J. Wyngaard, Y. Izumi, and O. R. Coté, Quart. J. Royal Meteorol. Soc. **98**, 563 (1972).
14. J. C. Kaimal and J. J. Finnigan, *Atmospheric Boundary Layer Flows* (Oxford, 1994).
15. S. Liu, H. Liu, M. Xu, M. Y. Leclerc, and T. Zhu, Boundary-Layer Meteorol. **98**, 83 (2001).
16. L. D. Landau and E. M. Lifshits, *Theoretical Physics, vol. 6, Fluid Mechanics* (Pergamon, London, 1987).
17. S. S. Zilitinkevich, T. Elperin, N. Kleeorin, I. Rogachevskii, and I. Esau, Boundary-Layer Meteorol. **146**, 341 (2013).