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ON ESTIMATES OF MEROMORPHIC FUNCTIONS AND SUMMATION OF SERIES IN THE ROOT VECTORS OF NONSELFADJOINT OPERATORS

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1°. Let L be an unbounded operator with discrete spectrum acting in a separable Hilbert space $\mathfrak{H}$. It is convenient in what follows to assume that $\ker L = 0$, i.e., L^{-1} exists and is a compact operator. Furthermore, it is assumed below that L^{-1} is an operator of finite order. Let $\{e_j\}$ be a sequence of eigenvectors and associated vectors (EV's and AV's) of L corresponding to the eigenvalues (EVAL's) $\{\lambda_j\}$, and for the adjoint operator let $\{e_j^*\}$ be the sequence of EV's and AV's biorthogonal to the system $\{e_j\}$.

We say that the EV's and AV's of L form a basis for the Abel summation method of order α, and write $L \in A(\alpha, \mathfrak{H})$, if the following conditions hold:

1) With the exception of finitely many EVAL's, the spectrum of L lies in the sector $|\arg\lambda| \leqslant \pi/2\theta$ for some $\theta > \alpha$.

2) There exist disjoint domains G_k $(k = 1, 2, \dots)$, each containing at most finitely many EVAL's λ_j, with the whole spectrum of L in their union and such that the series

$$(1)\qquad u(f,t) = \frac{-1}{2\pi i}\sum_{k=1}^{\infty} e^{-\lambda^\alpha t}(L-\lambda I)^{-1} f\, d\lambda$$

converges for any $t > 0$.

3) The limit $\lim_{t\to+0} u(f,t) = f$ exists in the sense of strong convergence in $\mathfrak{H}$.

Note that when there are no associated vectors, the series (1) coincides with the series

$$(1')\qquad u(f,t) = \sum_{k=1}^{\infty}\sum_{\lambda_j\in G_k} e^{-\lambda_j^\alpha t}(f, e_j^*)e_j,$$

and the choice of the domains G_k corresponds to an arrangement of parentheses in the Abel summation method.

Questions involving summation of series in the EV's and AV's of nonselfadjoint operators by the Abel method were first studied in [1]. These questions were later studied by many authors for abstract, differential and pseudodifferential operators; in particular, see [2]–[6].

2°. We shall use the following notation:

1) $\Lambda_\theta^\varphi = \{\lambda\colon |\varphi - \arg\lambda| \leqslant \pi/2\theta\}$, $1/2 < \theta < \infty$, and $\Lambda_\theta^0 = \Lambda_\theta$;

2) $P_{q,h}^0 = P_{q,h} = \{\lambda\colon \operatorname{Re}\lambda \geqslant 1, |\operatorname{Im}\lambda| \leqslant h(\operatorname{Re}\lambda)^q\}$ and $P_{q,h}^\varphi = \{\lambda\colon e^{-i\varphi}\lambda \in P_{q,h}\}$, $-\infty < q < 1$, $h > 0$
(domains bounded by parabolas, straight lines or hyperbolas, depending on the sign of q);

3) $K(\lambda_j, R) = \{\lambda\colon |\lambda - \lambda_j| \leqslant R\}$, and $K_R = K(0, R)$;

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4) $n_L(r) = \sum_{|\lambda_j| \leq r} 1$ and $\hat{n}_L(r) = \sum_{s_j^{-1} \leq r} 1$, where the λ_j are the EVAL's of L, and the s_j are the s-numbers of L^{-1};

5) $d(\lambda, G)$ is the distance from a point λ to the domain G; and

6) $W_2^k[0,1]$ is the Sobolev space of functions on $[0,1]$, with norm $\|f\|_{W_2^k} = \|f\|_{L_2} + \|f^{(k)}\|_{L_2}$.

The next result follows from [1] when a remark in [5] (see §36.4) is taken into account.

THEOREM. *Suppose that the spectrum of L lies in the domain $G = K_R \cup \Lambda_\theta$ for some $\theta > p$, R is sufficiently large, the estimate*

$$\left\|(L - \lambda I)^{-1}\right\| \leq C d^{-1}(\lambda, G) \tag{2}$$

holds outside of G, and the condition $\hat{n}_L(r) \leq C r^p$ is satisfied.

Then $L \in A(\alpha, \mathfrak{H})$ for $\alpha \in (p, \theta)$. Moreover, the domains G_k in (1) can be chosen as the simply connected components of the set

$$\mathcal{K} = \bigcup_{j=1}^{\infty} K\left(\lambda_j, e^{-|\lambda_j|^{\alpha-p-\varepsilon}}\right), \qquad 0 < \varepsilon < \alpha - p,$$

i.e., only terms corresponding to exponentially close EVAL's λ_j are combined in parentheses.

In investigating solutions of differential equations with operator coefficients (see [1]) it is important to establish the property $L \in A(\alpha, \mathfrak{H})$ for $\alpha = 1$. In this connection it is important to determine general conditions under which it is possible to extend the interval for the order of summability. This note deals with this question and with the investigation of the limit case when the Abel summation method of a specific order does not work. Our main result is

THEOREM 1. *Suppose that there are finitely many rays $\gamma_j = re^{i\varphi_j}$, $j = 1, 2, \ldots, m$, $0 < r < \infty$, $|\arg \varphi_j| < \pi/2\theta$, such that the spectrum of L lies in the domain*

$$G = \bigcup_{j=1}^{m} P_{q,h}^{\varphi j} \cup K_R$$

(in the domain $G = \cup_{j=1}^m \Lambda_\beta^{\varphi j} \cup K_R$, $R = R(\beta)$, for arbitrarily large β), while the estimate (2) *holds outside of G. If*

$$\varlimsup_{r \to \infty} n_L(r) r^{-p} = a < \infty \tag{3}$$

and $p - (1 - q) < \theta$ ($p \leq \theta$), then $L \in A(\alpha, \mathfrak{H})$ for all α with $\max[0, p - (1 - q)] < \alpha \leq \theta$ *($p \leq \alpha \leq \theta$). Moreover, if the number a in condition* (3) *is equal to zero or the number h can be chosen arbitrarily small, then the value $\alpha = p - (1 - q)$ is allowed if $p - (1 - q) > 0$. If $n_L(r + r^q) - n_L(r) = O(r^{p_1})$ for some $p_1 \geq 0$, then for $\alpha > p_1$ the domains G_k in* (1) *can be chosen as the simply connected components of the set $\mathcal{K} = \cup_{j=1}^{\infty} K(\lambda_j, e^{-|\lambda_j|^{\alpha-p-\varepsilon}})$.*

REMARK 1. If $L = L_0 + L_1$, where L_0 is a selfadjoint operator, $D(L_0) \subset D(L_1)$, and $\|L_1 f\| \leq C\|L_0^q f\|$ for all $f \in D(L_0)$ for some $q < 1$, then it is easy to verify that the estimate (2) holds for the resolvent of L, where $G = K_R \cup P_{q,h}$ and R and h are some numbers. Therefore, setting $m = 1$ and $\varphi_1 = 0$ in Theorem 1, we get the summability results in [2], [4] and [5] and the summability theorem in [3] for operators of finite order. The question of the arrangement of the parentheses was not studied in [2]–[4], but in [5] the question of the frequency of an arrangement of parentheses was studied under the assumption that $n_L(r) = r^p + O(r^{p_1})$, $p_1 < p$, though the question of just which terms should be combined in parentheses was not considered.

To prove Theorem 1 we use new estimates for meromorphic functions; these estimates are of independent interest.

THEOREM 2. *Suppose that $F(\lambda)$ is the quotient of two holomorphic functions of finite order in the domain $P_{q,h+\varepsilon}$, $\varepsilon > 0$, and that its poles $\{\lambda_j\}$ lie in $P_{q,h}$. Suppose that*

$$\varlimsup_{r\to\infty} n_F(r) r^{-p} \leqslant a, \qquad n_F(r) = \sum_{|\lambda_j|\leqslant r} 1. \tag{4}$$

Then there is a sequence $r_1 < r_2 < \cdots < r_k \to \infty$ such that

$$|F(\lambda)| \leqslant C\exp\left(C_1 h(a+\delta)|\lambda|^{p-(1-q)}\right)$$

for $|\lambda| = r_k$ and $\lambda \in P_{q,h}$, where $\delta > 0$ is an arbitrary number, and the constants C and C_1 do not depend on λ, $a+\delta$, nor h if $0 < h < h_0$, h_0 a fixed number.

If the condition $n_F(r + r^q) - n_F(r) = O(r^{p_1})$ holds for some $p_1 \geqslant 0$ instead of condition (4), *then the estimate*

$$|F(\lambda)| \leqslant C\exp|\lambda|^{\alpha}$$

holds outside the set $\mathcal{K} = \bigcup_{j=1}^{\infty} K(\lambda_j, \exp\{-|\lambda_j|^{\alpha-p_1-\varepsilon}\})$, $0 < \varepsilon < \alpha - p_1$, for $\alpha > p_1$.

3°. If $L \in A(1, \mathfrak{H})$, then a solution of the Cauchy problem

$$du/dt + Lu = 0, \qquad t > 0, \qquad u|_{t=0} = f \in \mathfrak{H} \tag{5}$$

(the last condition is understood in the sense that $u(t) \to f$ as $t \to +0$) exists and can be represented for all $t > 0$ by a convergent series in the EV's and AV's of L:

$$u(x,t) = \sum_{k=1}^{\infty} \sum_{j=N_k}^{N_{k+1}} c_j(t)(f, e_j^*)e_j, \tag{6}$$

where the coefficients $c_j(t)$ can be determined from (1) with $\alpha = 1$, and have the form $c_j(t) = \exp\{-\lambda_j t\}$ when the EVAL's are simple.

The existence of a solution of (5) can be guaranteed if the spectrum of L lies in the domain $G = K_R \cup \Lambda_\theta$ for some $\theta > 1$, while the estimate (2) holds outside this domain. In this case the integral

$$u(f,t) = -\frac{1}{2\pi i}\int_\Gamma e^{-\lambda t}(L - \lambda I)^{-1} f\, d\lambda, \tag{7}$$

where the contour Γ encircles the spectrum of L and is asymptotically directed along the sides of the angle $\Lambda_{\theta'}$, $1 < \theta' < \theta$, is a solution of problem (5) for $t > 0$ (see [1]), and use the remark on p. 347 of [5] enables us to show that $u(f,t) \to f$ as $t \to +0$. We note also that the function $u(f,t)$ is analytic in t in the sector $\Lambda_{\theta/(\theta-1)}$.

For certain concrete operators it has not been possible to show that the integral in (7) is representable for all $t > 0$ by a convergent series (6) in the EV's and AV's of L, but it has been possible to prove such a representation for sufficiently large t. The following definition is useful in this connection.

DEFINITION. We write $L \in H(\alpha, \mathfrak{H})$ if the following conditions hold:

1) All but finitely many of the EVAL's of L lie in a sector Λ_θ, $\theta > \alpha$.

2) For a suitable choice of the domains G_k the series (1) converges for all $t > t_0$, where $t_0 \geqslant 0$ is a sufficiently large number not depending on f, and the function $u(f,t)$ representing this series is analytic in some neighborhood of the ray (t_0, ∞).

3) $u(f,t)$ admits an analytic extension to a domain containing the ray $(0,\infty)$, and the limit $\lim_{t\to+0} u(f,t) = f$ exists in the strong sense.

We mention the following simple fact. If $L \in H(\alpha, \mathfrak{H})$, then the system $\{e_j\}$ of EV's and AV's of L and the system $\{e_j^*\}$ are complete in $\mathfrak{H}$. Indeed, if an element ψ is orthogonal to all the EV's and AV's of L, then the function $F(t) = (u(f, t), \psi)$ is equal to zero for $t > t_0$, so $F(t) \equiv 0$ in some domain containing the ray $(0, \infty)$. But $F(t) \to (f, \psi)$ as $t \to +0$; therefore, $(f, \psi) = 0$ for all $f \in \mathfrak{H}$, which implies that $\psi = 0$. Similarly, if f is orthogonal to the elements $\{e_j^*\}$, then $(f, \psi) = 0$ for all $\psi \in \mathfrak{H}$, and so $f = 0$.

If $L \in H(1, \mathfrak{H})$, then a solution of problem (5) exists and can be represented by a convergent series (6) for large $t > t_0$. Indeed, in the case $L \in H(1, \mathfrak{H})$ the function $u(x, t)$ defined for $t > t_0$ by (6) satisfies the equation $u(f, t) = -L^{-1}u'(f, t)$ for $t > t_0$, and the functions $u(x, t)$ and $u'(x, t)$ can be extended analytically to a neighborhood of the ray $(0, \infty)$. Hence, $u'(f, t) \in \mathfrak{H}$ for all $t > 0$, and the uniqueness of the analytic extension gives us that $u(f, t) \subset \mathfrak{D}(L)$ for all $t > 0$; moreover, $u(f, t) \to f$ as $t \to 0$. Therefore, $u(x, t)$ is a solution of (5).

Using Theorem 2, we get the following result.

THEOREM 3. *If* (3) *holds for some* $p > 0$, *the spectrum of* L *lies in the domain* $G = K_R \cup P_{q,h}$, $p - (1 - q) > 0$ $(G = \Lambda_\theta \cup K_R, \theta > p)$, *and the estimate* (2) *holds outside* G, *then* $L \in H(p - (1 - q), \mathfrak{H})$ $(L \in H(p, \mathfrak{H}))$.

4°. We present two examples in which the limit situation considered in the preceding section is encountered.

EXAMPLE 1. Consider the operator

$$Lu = \Delta u + Tu, \qquad u|_{\partial\Omega} = 0, \tag{8}$$

in a bounded domain $\Omega \subset R^3$, where Δ is the Laplace operator, and T is a differential operator of order 1 whose coefficients are bounded functions.

It is known (see, for example, [1]) that $n(L, r) \sim Cr^{3/2}$, and by using Remark 1 we can get the estimate (2) for the resolvent of L outside the domain $G = K_R \cup P_{q,h}$ for $q = 1/2$. We then conclude from Theorem 3 that $L \in H(1, L_2(\Omega))$. Thus, problem (5) with L defined in (8) has a solution which is given by (7) and can be represented by a convergent series in the EV's and AV's of L for large t.

EXAMPLE 2. Consider the problem

$$-\frac{\partial^2 u}{\partial t^2} - 2a\frac{\partial^2 u}{\partial t \partial x} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad -1 < a < 1, \tag{9}$$

in the half-strip $S = \{0 \leqslant x \leqslant 1, 0 < t < \infty\}$, with the boundary conditions

$$u(0, t) = u(1, t) = 0, \qquad t > 0, \qquad u(x, t)|_{t=0} = f(x), \qquad u(x, \infty) = 0. \tag{10}$$

With this problem we associate the operator pencil $M(\lambda) = A + i\lambda B - \lambda^2 I$, where $A = -d^2/dx^2$, $B = 2ai\,d/dx$, I is the identity operator, and the domain of the operator $M(\lambda)$ is $\mathfrak{D}(M) = \{y(x)\colon y(x) \in W_2^2[0, 1], y(0) = y(1) = 0\}$. The operator pencil $M_1(\lambda) = A^{-1/2}M(\lambda)A^{-1/2}$ (see [7]) admits the factorization

$$M_1(\lambda) = (I + \lambda A^{-1/2}K^{-1})(I - \lambda K A^{-1/2})$$

with respect to the imaginary axis. Then we get

$$M(\lambda) = (A^{1/2}K + \lambda I)(K^{-1}A^{1/2} - \lambda I),$$

and the EVAL's and EV's of the operator $L = K^{-1}A^{1/2}$ coincide with the EVAL's and EV's of the pencil $M(\lambda)$ in the right half-plane, which in this case have the explicit form $\lambda_k = k$, $e_k(x) = e^{\beta k x} \sin kx$, $\beta = a/\sqrt{1 - a^2}$. Using the estimate $\|M^{-1}(\lambda)\| \leqslant C|\lambda|^{-2}$ in the angles abutting on the imaginary axis (see [7]), we can get the estimate $\|(L-\lambda I)^{-1}\| \leqslant C|\lambda|^{-1}$ for $\lambda \notin \Lambda_\theta$, $1 < \theta < \pi/(\pi - 2\arccos a)$. Then $L \in H(1, L_2)$, the integral (7) gives a solution of problem (9), (10), and for large t this solution can be represented by a convergent series

$$u(f, t) = \sum_{k=1}^{\infty} c_k e^{-kt} e^{\beta kx} \sin kx.$$

It can be shown that this series does not, generally speaking, converge for all $t > 0$, i.e., $L \notin A(1, L_2)$.

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