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CLASSICAL HIGGS FIELDS

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We consider a classical gauge theory on a principal fiber bundle $P \to X$ in the case where its structure group G is reduced to a subgroup H in the presence of classical Higgs fields described by global sections of the quotient fiber bundle $P/H \to X$. We show that matter fields with the exact symmetry group H in such a theory are described by sections of the composition fiber bundle $Y \to P/H \to X$, where $Y \to P/H$ is the fiber bundle with the structure group H, and the Lagrangian of these sections is factored by virtue of the vertical covariant differential determined by a connection on the fiber bundle $Y \to P/H$.

Keywords: gauge field, Higgs field, matter field, fiber bundle, connection

This paper is dedicated to the 75th birthday of Academician of the RAS A. A. Slavnov

1. Introduction

In the general case, a classical gauge theory comprises fields of three types: gauge potentials, matter fields, and classical Higgs fields. The last appear in a gauge theory after spontaneous symmetry breaking that results in the transformation group for matter fields becoming smaller than the gauge group. We note that the nature of the phenomenon of spontaneous symmetry breaking is more general; for example, in quantum field theory, it is characterized by a Higgs vacuum similar to a background classical field [1].

We can completely formulate a classical field theory in terms of a Lagrangian theory on smooth fiber bundles whose sections are classical fields [2], [3]. Correspondingly, a classical gauge theory is a classical field theory on principal and associated fiber bundles [3], [4]. In a gauge theory on a principal fiber bundle $P \to X$, a spontaneous symmetry breaking pertains to reducing its structure Lie group G to a closed subgroup H of exact symmetries [5]–[8]. Such a reduction is possible if and only if the quotient fiber bundle $P/H \to X$ admits global sections h (Theorem 1). These sections can be interpreted as classical Higgs fields [5], [6], [8], [9]. They parameterize the principal reduced subbundles P^h (with the structure group H) of the principal fiber bundle P. These subbundles are not equivalent (Remark 3) and are nonisomorphic in the general case (Theorem 4).

If we reduce the structure group G of a principal fiber bundle $P \to X$ to a closed subgroup H, then in the framework of this gauge theory, we can represent matter fields with the exact symmetry group H as sections s_h associated with subbundles $P^h \subset P$ of fiber bundles Y^h with typical fibers V endowed with the left action of the group H. Because the subbundles P^h are not equivalent, such a matter field can enter the theory only together with a definite Higgs field h. The problem of describing the set of all pairs (s_h, h) of matter and Higgs fields thus arises. These pairs are sections of the composition fiber bundle $Y \to P/H \to X$ (see (21)), where $Y \to P/H$ is a fiber bundle with the structure group H and a typical fiber V, and this fiber bundle is associated with the principal H-bundle $P \to P/H$ (Sec. 5). The geometry of such composition fiber bundles was studied in [3], [9]. The key observation is that for any section h of

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the quotient fiber bundle $P/H \to X$, the fiber bundle h^*Y induced by the fiber bundle $Y \to P/H$ is a subbundle Y^h of $Y \to X$, and this subbundle is associated with the principal reduced subbundle $P^h \subset P$ with the structure group H. Its sections s_h correspond to matter fields in the presence of a background Higgs field h.

Following [9], we here prove that the composition fiber bundle $Y \to X$ is a P-associated fiber bundle with the structure group G (Theorem 9). This allows describing matter fields with the symmetry group H in terms of the gauge theory on the principal fiber bundle P (Sec. 6). The key point is that the Lagrangian of these matter fields is factored through the vertical covariant differential $\widetilde{D}$ (see (29)) determined by an H-connection on the fiber bundle $Y \to P/H$. The restriction A_h of this connection on the subbundle $Y^h \subset Y$ then becomes an H-connection on this subbundle (Proposition 3), and the restriction Y^h of the vertical covariant differential $\widetilde{D}$ then becomes the differential covariant with respect to the connection A_h (Proposition 4).

But the problem is that a connection on a fiber bundle $Y \to P/H$ is not a dynamical variable in a gauge theory. We therefore assume that the Lie algebra of the group G admits Cartan decomposition (18). We here show that in this case, any G-connection on the principal fiber bundle $P \to X$ induces an H-connection on any reduced subbundle P^h (Theorem 7) and therefore induces the desired H-connection on the fiber bundle $Y \to P/H$ (Theorem 10). On configuration space (40), this results in the gauge theory of gauge potentials of the group G, of matter fields with the symmetry subgroup $H \subset G$, and of classical Higgs fields.

An example of a classical Higgs field is the metric gravitational field in the gauge theory of gravity with spontaneous symmetry breaking in the presence of Dirac spinor fields with the Lorentz group of symmetries or by the geometric equivalence principle [5], [10], [11].

2. The gauge theory on principal fiber bundles

We consider smooth fiber bundles (of the class C^∞). We assume that smooth manifolds are separable, locally compact, countable at infinity, paracompact topological spaces.

As already mentioned, we formulate the classical gauge theory on the principal fiber bundle

$$\pi_P \colon P \to X \tag{1}$$

on an n-dimensional manifold X with the structure Lie group G acting on P from the right fiberwise freely and transitively [3], [4]. For brevity, we call P the principal G-bundle. Its atlas

$$\Psi_P = \{(U_\alpha, z_\alpha)\varrho_{\alpha\beta}\} \tag{2}$$

is defined by a family of local sections z_α with G-valued transition functions $\varrho_{\alpha\beta}$ such that $z_\beta(x) = z_\alpha(x)\rho_{\alpha\beta}(x)$, $x \in U_\alpha \cap U_\beta$.

Because G acts on P from the right, we consider the fiber bundles

$$T_G P = TP/G, \qquad V_G P = VP/G \tag{3}$$

over X. A typical fiber of the bundle $V_G P \to X$ is then the right Lie algebra $\mathfrak{g}_r$ of the group G with the basis $\{\varepsilon_p\}$ on which G acts in the adjoint representation. Sections of the fiber bundles $T_G P \to X$ and $V_G P \to X$ in (3) are the respective G-invariant vector fields and vertical G-invariant vector fields on P.

In the general setting, a connection on a fiber bundle $P \to X$ is defined as a section of the fiber bundle $J^1P \to P$, where J^1P is the jet manifold of the fiber bundle $P \to X$ [4], [12]. Because we assume that connections on the principal fiber bundle are equivariant with respect to the structure group action (for brevity, we call them G-connections), they are global sections of the quotient bundle of connections

$$C = J^1P/G \to X. \tag{4}$$

This is an affine fiber bundle modeled over the vector bundle $T^*X \otimes_X V_G P$. Because of the canonical embedding

$$C \underset{X}{\longrightarrow} dx^\mu \otimes (\partial_\mu + a^p_\mu e_p) \in T^*X \underset{X}{\otimes} T_G P,$$

G-connections on P can also be represented in terms of $T_G P$-valued forms

$$A = dx^\lambda \otimes (\partial_\lambda + A^p_\lambda e_p). \tag{5}$$

Let V be a manifold admitting a left action of the stricture group G of the fiber bundle P in (1). The bundle associated with P and having a typical fiber V is then defined as a quotient space

$$Y = (P \times V)/G, \qquad (p, v)/G = (pg, g^{-1}v)/G, \quad g \in G. \tag{6}$$

For brevity, we call it a P-associated fiber bundle.

Every atlas Ψ_P given by (2) of the principal fiber bundle P determines the atlas

$$\Psi_Y = \{(U_\alpha, \psi_\alpha)\}, \qquad \psi_\alpha(x)\colon (z_\alpha(x), v)/G \to v, \tag{7}$$

of the associated fiber bundle Y in (6) and endows Y with fiberwise coordinates (x^λ, y^i).

Every G-connection A given by (5) on the principal fiber bundle P determines the connection

$$A = dx^\lambda \otimes (\partial_\lambda + A^p_\lambda I^i_p \partial_i) \tag{8}$$

on the associated fiber bundle Y, where $\{I_p\}$ is a representation of the Lie algebra $\mathfrak{g}_r$ in its typical fiber V.

3. Reduced structures and Higgs fields

Let H, $\dim H > 0$, be a closed subgroup of the structure group G (see Remark 1 below). Then the composition fiber bundle

$$P \to P/H \to X \tag{9}$$

is determined, where

$$P_\Sigma = P \xrightarrow{\pi_{P\Sigma}} P/H \tag{10}$$

is the principal fiber bundle with the structure group H and

$$\Sigma = P/H \xrightarrow{\pi_{\Sigma X}} X \tag{11}$$

is a P-associated fiber bundle with the typical fiber G/H on which the structure group G acts from the left.

Remark 1. A closed subgroup H of a Lie group G is a Lie group. We consider the quotient space G/H of the group G with respect to the right action of H on G. We can show that

$$\pi_{GH}\colon G \to G/H \tag{12}$$

is the principal fiber bundle with the structure group H [13]. In particular, if H is a maximum compact subgroup of G, then the quotient space G/H is diffeomorphic to $\mathbb{R}^m$, and fiber bundle (12) is then trivial.

The structure Lie group G of the principal fiber bundle P is said to be reduced to its closed subgroup H if the following equivalent conditions are satisfied:

1. The principal fiber bundle P admits atlas (2) with H-valued transition functions $\varrho_{\alpha\beta}$.
2. We have a principal reduced subbundle P_H of the fiber bundle P with the structure group H.

Indeed, if $P_H \subset P$ is a reduced subbundle, then its atlas (2) generated by local sections z_α is also an atlas of the fiber bundle P with H-valued transition functions. Conversely, let (2) be an atlas of the fiber bundle P with H-valued transition functions $\varrho_{\alpha\beta}$. For any $x \in U_\alpha \subset X$, we define a submanifold $z_\alpha(x)H \subset P_x$. These submanifolds constitute an H-subbundle of P because $z_\alpha(x)H = z_\beta(x)H\varrho_{\beta\alpha}(x)$ on the intersections $U_\alpha \cap U_\beta$.

Remark 2. Principal reduced H-subbundles of the principal G-bundle are sometimes called G-structures [8], [14]–[16]. In [14], [16], only reduced structures of the principal fiber bundle LX of linear frames in tangent spaces TX to the manifold X were considered, and the isomorphism class of these structures was confined to holonomy automorphisms of LX, i.e., to functorial liftings to LX of diffeomorphisms of the base X. The notion of G-structure was extended to an arbitrary fiber bundle in [15], where it was interpreted as the Klein–Chern geometry. In the case where the Lie algebra of a group G admits Cartan decomposition (18), the G-structure is said to be reduced [17] and manifests several additional features (Theorem 7).

The key fact is the following statement [18].

Theorem 1. *We have a one-to-one correspondence*

$$P^h = \pi^{-1}_{P\Sigma}(h(X)) \tag{13}$$

between principal reduced H-subbundles $i_h\colon P^h \to P$ of the fiber bundle P and global sections h of the quotient bundle $P/H \to X$ given by (11).

Formula (13) implies that the principal reduced H-subbundle P^h is a reduction h^*P_Σ of the principal H-bundle P_Σ in (10) on a submanifold $h(X) \subset \Sigma$. At the same time, every atlas Ψ_h of the fiber bundle P^h generated by a family of its local sections is simultaneously an atlas of the principal G-bundle P and an atlas of the P-associated fiber bundle $\Sigma \to X$ given by (11) with H-valued transition functions. Relative to the atlas Ψ_h of a fiber bundle Σ, the global section h of this bundle then takes values in the kernel of the quotient space G/H.

As already mentioned, we interpret global sections of the quotient bundle $P/H \to X$ as classical Higgs fields in classical gauge theory [3], [8], [9].

For example, we can formulate a theory of gravity on an oriented four-dimensional manifold X as a gauge theory on the principal fiber bundle LX of linear frames tangent to X with the structure group $GL^+(4,\mathbb{R})$ reduced to the Lorentz group $SO(1,3)$ [5], [10], [11]. Global sections of the corresponding quotient bundle $LX/SO(1,3)$ are pseudo-Riemannian metrics on the manifold X, which are identified with gravitational fields in general relativity.

Reducing a structure group is not always possible. In particular, in the above case of the gauge theory of gravity, it occurs on noncompact manifolds X and on compact manifolds with the zero Euler characteristic. We note the following fact [13].

Theorem 2. *A bundle $Y \to X$ whose typical fiber is diffeomorphic to the manifold $\mathbb{R}^m$ always admits a global section, and every section of it over a closed submanifold of the base X can be extended, albeit nonuniquely, to the global section.*

Corollary 1. *The structure group G of the principal fiber bundle P is reducible to a closed subgroup H if the quotient space G/H is diffeomorphic to the Euclidean space $\mathbb{R}^m$.*

In particular, we always have a reduction of the Lie structure group G to its maximum compact subgroup H (see Remark 1). This includes the cases $G = GL(m,\mathbb{C})$, $H = U(m)$ and $G = GL(n,\mathbb{R})$, $H = O(n)$, which are important for applications.

We also note that different principal H-subbundles P^h and $P^{h'}$ of the principal G-bundle P are not necessarily mutually isomorphic.

Theorem 3 [13]**.** *If the quotient space G/H is diffeomorphic to the Euclidean space $\mathbb{R}^m$, then all principal reduced H-subbundles of the principal G-bundle P are mutually isomorphic.*

Theorem 4. *Let the Lie structure group G of the principal fiber bundle P be reduced to its closed subgroup H. We have the following statements*:

1. *Every vertical automorphism Φ of the principal fiber bundle P maps its principal reduced H-subbundle P^h to the isomorphic principal reduced H-subbundle $P^{h'} = \Phi(P^h)$.*
2. *Conversely, let two reduced subbundles P^h and $P^{h'}$ of the principal fiber bundle $P \to X$ be mutually isomorphic, and let $\Phi\colon P^h \to P^{h'}$ be their isomorphism over X. We can then extend Φ to the isomorphism of the whole fiber bundle P.*

Proof. Let

$$\Psi^h = \{(U_\alpha, z^h_\alpha), \varrho^h_{\alpha\beta}\} \tag{14}$$

be an atlas of a principal reduced subbundle P^h, where z^h_α are local sections $P^h \to X$ and $\varrho^h_{\alpha\beta}$ are the transition functions. Given a vertical automorphism Φ of the fiber bundle P, we can endow the subbundle $P^{h'} = \Phi(P^h)$ with the atlas

$$\Psi^{h'} = \{(U_\alpha, z^{h'}_\alpha), \varrho^{h'}_{\alpha\beta}\} \tag{15}$$

determined by its local sections $z^{h'}_\alpha = \Phi \circ z^h_\alpha$. We can then easily obtain

$$\varrho^{h'}_{\alpha\beta}(x) = \varrho^h_{\alpha\beta}(x), \quad x \in U_\alpha \cap U_\beta, \tag{16}$$

i.e., that transition functions of atlas (15) take values in the subgroup H. Conversely, every automorphism $(\Phi, \mathrm{Id}\, X)$ of principal reduced subbundles P^h and $P^{h'}$ of the fiber bundle P determines an H-equivariant G-valued function f on P^h by the relation $pf(p) = \Phi(p)$, $p \in P^h$. Its extension to a G-equivariant function on P is defined as

$$f(pg) = g^{-1} f(p) g, \quad p \in P^h, \quad g \in G.$$

The relation $\Phi_P(p) = pf(p)$, $p \in P$, then defines a vertical automorphism Φ_P of the fiber bundle P whose restriction on P^h coincides with Φ.

Remark 3. In Theorem 4, we can regard the principal G-bundle P endowed with the atlas Ψ^h given by (14) as a P^h-associated fiber bundle with the structure group H acting on its typical fiber G from the left. Correspondingly, being equipped with the atlas $\Psi^{h'}$ given by (15), the bundle P is a $P^{h'}$-associated H-bundle. The H-bundles (P, Ψ^h) and $(P, \Psi^{h'})$ are not equivalent then, because their atlases Ψ^h and $\Psi^{h'}$ are not equivalent. Indeed, the union of these atlases is the atlas

$$\Psi = \{(U_\alpha, z^h_\alpha, z^{h'}_\alpha), \varrho^h_{\alpha\beta}, \varrho^{h'}_{\alpha\beta}, \varrho_{\alpha\alpha} = f(z_\alpha)\}$$

with the transition functions

$$\varrho_{\alpha\alpha}(x) = f(z_\alpha(x)), \qquad z_\alpha^{h'}(x) = z_\alpha^h(x)\varrho_{\alpha\alpha}(x) = (\Phi_P \circ z_\alpha^h)(x)$$

between the corresponding maps (U_α, z_α^h) and $(U_\alpha, z_\alpha^{h'})$ from the respective atlases Ψ^h and $\Psi^{h'}$. But transition functions $\varrho_{\alpha\alpha}$ are not H-valued. At the same time, equality (16) implies that transition functions of both atlases constitute the same cocycle. Hence, the H-bundles (P, Ψ^h) and $(P, \Psi^{h'})$ are associated. Because of the isomorphism $\Phi\colon P^h \to P^{h'}$, we can write

$$P = (P^h \times G)/H = (P^{h'} \times G)/H, \qquad (p \times g)/H = (\Phi(p) \times f^{-1}(p)g)/H.$$

For any $\rho \in H$ we then obtain

$$(p\rho, g)/H = (\Phi(p)\rho, f^{-1}(p)g)/H = (\Phi(p), \rho f^{-1}(p)g)/H = (\Phi(p), f^{-1}(p)\rho' g)/H,$$

where

$$\rho' = f(p)\rho f^{-1}(p). \tag{17}$$

Hence, we can treat $(P, \Psi^{h'})$ as a P^h-associated bundle with the same typical fiber G as for (P, Ψ^h), but the action $g \to \rho' g$ in (17) of the structure group H on a typical fiber of the bundle $(P, \Psi^{h'})$ is not equivalent to its action $g \to \rho g$ on a typical fiber of the bundle (P, Ψ^h), because they have different orbits in G.

4. Reductions of connections

We present compatibility conditions for connections on the principal fiber bundle with its reduction structures [4], [18].

Theorem 5. *Because connections on the principal fiber bundle are equivariant, every H-connection A_h on an H-subbundle P^h of the principal G-bundle P is extendible to a G-connection on P.*

Theorem 6. *Conversely, the connection A given by (5) on the principal G-bundle P is reducible to an H-connection on the principal reduced H-subbundle P^h of the fiber bundle P if and only if the corresponding global section h of the fiber bundle $P/H \to X$ associated with P is an integral section of the associated connection A given by (8) on $P/H \to X$, i.e., $D^A h = 0$, where D^A is the covariant differential determined by this connection.*

In particular, a connection on P is ~~not~~ always reducible to a connection on P^h under the following condition [3], [18].

Theorem 7. *Let the Lie algebra $\mathfrak{g}$ of a Lie group G be a direct sum*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f} \tag{18}$$

of the Lie algebra $\mathfrak{h}$ of a Lie group H and its complement $\mathfrak{f}$ such that we have the condition $[\mathfrak{h}, \mathfrak{f}] \subset \mathfrak{f}$ for the commutation relations. Let A be a G-connection on the principal fiber bundle P. We consider the principal reduced fiber bundle P^h with an atlas Ψ_h, which is simultaneously an atlas of the fiber bundle P. Then the induction $\bar{A}_h = h^* A_\mathfrak{h}$ on P^h of the $\mathfrak{h}$-valued constituent $A_\mathfrak{h}$ of the connection form A given by (5) written in terms of the atlas Ψ_h is an H-connection on P^h.

In this case, matter fields with an exact symmetry group can be written in the presence of gauge fields with a larger group of spontaneously broken symmetries (see Sec. 6).

In particular, decomposition (18) occurs if H is the Cartan subgroup of G, and we therefore call this the Cartan decomposition.

For example, in the gauge theory of gravity on the principal frame bundle LX of the Lie algebra, the groups $\mathfrak{g} = gl(4, \mathbb{R})$ and $\mathfrak{h} = so(1,3)$ satisfy condition (18), and for a given pseudo-Riemannian metric h, a general linear connection can be decomposed into the sum of Christoffel symbols, the contorsion tensor, and the nonmetricity tensor. Its first two terms constitute the Lorentzian connection on the reduced $SO(1.3)$-subbundle $L^hX \subset LX$, which allows describing Dirac spinor fields in the theory of gravity in the presence of a general linear connection [10], [11], [19].

We can also generate connections on principal reduced subbundles in another way.

Let $P \to X$ be the principal fiber bundle. For a morphism of manifolds $\phi\colon X' \to X$, the induced fiber bundle $\phi^*P \to X'$ is the principal fiber bundle with the same structure group as for P. If A is a connection on the principal fiber bundle P, then the induced connection ϕ^*A on ϕ^*P is a connection on it as on a principal fiber bundle [4]. We hence obtain a result important in what follows [3], [4].

Theorem 8. *We consider composition fiber bundle* (9). *Let A_Σ be a connection on the principal H-bundle $P \to \Sigma$ given by* (10). *Then for any principal reduced H-bundle $i_h\colon P^h \to P$, the induced connection $i_h^* A_\Sigma$ on P^h is an H-connection on this bundle.*

We note that the Lagrangian of matter fields, as already mentioned in Sec. 1, is factored through the vertical covariant differential determined just by a connection on the fiber bundle $P \to P/H$ (see Sec. 6).

5. Associated bundles and matter fields

By virtue of Theorem 1, we have a one-to-one correspondence between principal reduced H-subbundles P^h of the principal fiber bundle P and Higgs fields h. For a given such subbundle P^h, we introduce the notation

$$Y^h = (P^h \times V)/H \tag{19}$$

for the associated vector bundle with the typical fiber V admitting the left action of the exact symmetry group H. Its sections s_h describe matter fields in the presence of the Higgs field h and an H-connection A_h on the principal fiber bundle P^h.

Difference fiber bundles Y^h and $Y^{h' \neq h}$ (19) are mutually related as follows. If the principal reduced H-subbundles P^h and $P^{h'}$ of the principal G-bundle P are isomorphic by virtue of Theorem 4, then the P^h-associated fiber bundle Y^h given by (19) is also associated as

$$Y^h = (\Phi(p) \times V)/H \tag{20}$$

to the subbundle $P^{h'}$. If its typical fiber V admits the action of the entire group G, then the P^h-associated fiber bundle Y^h in (19) is also P-associated,

$$Y^h = (P^h \times V)/H = (P \times V)/G.$$

Such P-associated fiber bundles are equivalent as G-bundles but not equivalent as H-bundles, because transition functions between their atlases are not H-valued (see Remark 3).

For example, in the gauge theory of gravity on a manifold X, the tangent bundle TX treated for a given pseudo-Riemannian metric h as a L^hX-associated bundle is a fibering into copies of the Minkowski space

$M^h X$. But for different pseudo-Riemannian metrics h and h', the fiber bundles $M^h X$ and $M^{h'} X$ are not equivalent; in particular, representations of their elements in terms of γ-matrices are not equivalent [11], [20].

Because different fiber bundles Y^h and $Y^{h'\neq h}$ are not equivalent in a definite sense and are not associated in general, we can consider a V-valued matter field only in pair with a definite Higgs field. We therefore encounter the problem of characterization of the set of all pairs (s_h, h) of the matter and Higgs fields.

To describe a matter field in the presence of different Higgs fields, we consider composition fiber bundle (9) and the composition fiber bundle

$$Y \to^{\pi_{Y\Sigma}} \Sigma \to^{\pi_{\Sigma X}} X, \tag{21}$$

where $Y \to \Sigma$ is a P_Σ-associated bundle

$$Y = (P \times V)/H \tag{22}$$

with the structure group H. For a given global section h of the fiber bundle $\Sigma \to X$ given by (11) and for the corresponding principal reduced H-bundle $P^h = h^* P$, fiber bundle (19) associated with P^h is a restriction

$$Y^h = h^* Y = (h^* P \times V)/H \tag{23}$$

of the fiber bundle $Y \to \Sigma$ on $h(X) \subset \Sigma$.

We can then prove the following statement [3], [9].

Proposition 1. *Every global section s_h of the fiber bundle Y^h given by (23) is a global section of composition fiber bundle (21) projected onto the section $h = \pi_{Y\Sigma} \circ s$ of the fiber bundle $\Sigma \to X$. Conversely, any global section s of the composition fiber bundle $Y \to X$ given by (21), when projected onto the section $h = \pi_{Y\Sigma} \circ s$ of the fiber bundle $\Sigma \to X$, takes values in the subbundle $Y^h Y$ given by (23). Hence, we have a one-to-one correspondence between sections of the fiber bundle Y^h given by (19) and those of composition bundle (21) that cover h.*

Proposition 2. *An atlas*

$$\Psi_{P\Sigma} = \{(U_{\Sigma\iota}, z_\iota), \varrho_{\iota\kappa}\} \tag{24}$$

of the principal H-bundle $P \to \Sigma$ and correspondingly of the associated fiber bundle $Y \to \Sigma$ determines the atlas

$$\Psi^h = \{(\pi_{P\Sigma}(U_{\Sigma\iota}), z_\iota \circ h), \varrho_{\iota\kappa} \circ h\} \tag{25}$$

of the reduced H-subbundle P^h and hence of the bundle Y^h. Atlas (25) is simultaneously an atlas of P with H-valued transition functions.

Given an atlas Ψ_P of the principal fiber bundle P, which determines the atlas of the associated fiber bundle $\Sigma \to X$ in (11), and an atlas $\Psi_{Y\Sigma}$ of the fiber bundle $Y \to \Sigma$, we can endow the composition fiber bundle $Y \to X$ in (21) with the corresponding coordinate system $(x^\lambda, \sigma^m, y^i)$, where (σ^m) are fiberwise coordinates on $\Sigma \to X$ and (y^i) are those on $Y \to \Sigma$.

Proposition 3. *Let*

$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + \mathcal{A}^a_\lambda e_a) + d\sigma^m \otimes (\partial_m + \mathcal{A}^a_m e_a) \tag{26}$$

be a connection on the principal H-bundle $P \to \Sigma$, and let

$$A_{Y\Sigma} = dx^\lambda \otimes (\partial_\lambda + \mathcal{A}^a_\lambda(x^\mu, \sigma^k) I^i_a \partial_i) + d\sigma^m \otimes (\partial_m + \mathcal{A}^a_m(x^\mu, \sigma^k) I^i_a \partial_i) \tag{27}$$

be the associated connection on $Y \to \Sigma$, where $\{I_a\}$ is a representation of the right Lie algebra $\mathfrak{h}_r$ of the group H in V. Then for any H-subbundle $Y^h \to X$ of the composition fiber bundle $Y \to X$, the induced connection

$$A_h = h^* A_{Y\Sigma} = dx^\lambda \otimes [\partial_\lambda + (\mathcal{A}^a_m(x^\mu, h^k)\partial_\lambda h^m + \mathcal{A}^a_\lambda(x^\mu, h^k))I^i_a \partial_i] \tag{28}$$

on Y^h is associated with the induced connection $h^* A_\Sigma$ on the principal reduced H-subbundle P^h in Theorem 8.

Every connection A_Σ given by (26) on the fiber bundle $Y \to \Sigma$ determines the first-order differential operator

$$\widetilde{D}\colon J^1 Y \to T^*X \underset{Y}{\otimes} V_\Sigma Y, \qquad \widetilde{D} = dx^\lambda \otimes (y^i_\lambda - \mathcal{A}^i_\lambda - \mathcal{A}^i_m \sigma^m_\lambda)\partial_i, \tag{29}$$

acting on the composition fiber bundle $Y \to X$, where $V_\Sigma Y$ is the vertical tangent bundle to the fiber bundle $Y \to \Sigma$. It is called the vertical covariant differential and has the following important property.

Proposition 4. *For any section h of a fiber bundle $\Sigma \to X$, the restriction of the vertical differential $\widetilde{D}$ given by (29) on the fiber bundle Y^h given by (23) coincides with the differential D^{A_h} on Y^h that is covariant with respect to the induced connection A_h given by (28).*

We thus find that those are the sections of the composition fiber bundle $Y \to X$ given by (21) that describe the pairs (s_h, h) of the matter and Higgs fields in a classical gauge theory with spontaneous symmetry breaking.

The following fact is essential when constructing a gauge theory with spontaneous symmetry breaking.

Theorem 9. *The composition fiber bundle $Y \to X$ given by (21) is a P-associated fiber bundle whose structure group is G and whose typical fiber is an H-bundle*

$$W = (G \times V)/H, \tag{30}$$

associated with the principal H-bundle $G \to G/H$ given by (12).

Proof. We represent the fiber bundle $P \to X$ as a P-associated fiber bundle

$$P = (P \times G)/G, \qquad (pg', g) = (p, g'g), \quad p \in P, \quad g, g' \in G,$$

whose typical fiber is the group space of G on which the group G acts by left multiplications. We can then represent quotient (22) in the form

$$Y = (P \times (G \times V)/H)/G,$$
$$(pg', (g\rho, v)) = (pg', (g, \rho v)) = (p, g'(g, \rho v)) = (p, (g'g, \rho v)).$$

Therefore, Y given by (22) is a P-associated bundle with the typical fiber W given by (30) on which the structure group G acts according to the law

$$g'\colon (G \times V)/H \to (g'G \times V)/H. \tag{31}$$

This is the so-called induced representation of a group G by its subgroup H [21]. Given an atlas $\{(U_a, z_a)\}$ of the principal H-bundle $G \to G/H$, induced representation (31) becomes

$$
\begin{aligned}
g'\colon (\sigma, v) = (z_a(\sigma), v)/H \;\to\; (\sigma', v') &= (g' z_a(\sigma), v)/H = \\
&= (z_b(\pi_{GH}(g' z_a(\sigma)))\rho', v)/H = \\
&= (z_b(\pi_{GH}(g' z_a(\sigma))), \rho' v)/H, \\
\rho' = z_b^{-1}(\pi_{GH}(g' z_a(\sigma))) g' z_a(\sigma) \in H, \quad \sigma \in U_a, \quad & \pi_{GH}(g' z_a(\sigma)) \in U_b.
\end{aligned}
$$

For example, if H is the Cartan subgroup of G, then induced representation (31) is a known nonlinear realization of the group G [3], [22], [23].

6. The Lagrangian of matter fields

Propositions 3 and 4 and Theorem 9 imply the following peculiarity of formulating a Lagrangian gauge theory with spontaneous symmetry breaking.

Let $P \to X$ be the principal fiber bundle whose structure group G is reduced to a closed subgroup H. Let Y be the P_Σ-associated fiber bundle (22). The total configuration space of the gauge theory of G-connections on P in the presence of matter and Higgs fields is

$$
J^1C \underset{X}{\times} J^1Y, \tag{32}
$$

where C is connection bundle (4) and J^1Y is the manifold of jets of the fiber bundle $Y \to X$. The total Lagrangian on configuration space (32) is the sum

$$
L_{\mathrm{tot}} = L_A + L_{\mathrm{m}} + L_\sigma \tag{33}
$$

of the gauge field Lagrangian L_A, the matter field Lagrangian L_{m}, and the Higgs field Lagrangian L_σ.

Because we do not specify the gauge and Higgs fields and because their Lagrangians can take very different forms depending on a model, for instance, in the gauge theory of gravity and in a theory of the Yang–Mills type, we here consider only the matter field Lagrangan L_{m}. By Proposition 4, it factors into

$$
L_{\mathrm{m}}\colon J^1C \underset{X}{\times} J^1Y \xrightarrow{\widetilde{D}} T^*X \underset{Y}{\otimes} V_\Sigma Y \to \overset{n}{\wedge} T^*X \tag{34}
$$

through the vertical differential $\widetilde{D}$ given by (29). Moreover, we can demonstrate that such a factorization is a necessary condition for the gauge invariance of L_{m} under automorphisms of the principal G-bundle $P \to X$ [3].

But the problem is that connection A_Σ given by (26) on the fiber bundle $Y \to P/H$, which determines $\widetilde{D}$, is not a dynamical variable in the gauge theory. We therefore assume that the Lie algebra of the group G admits Cartan decomposition (18). In this case, any G-connection A on the principal fiber bundle $P \to X$ determines the H-connection $\bar{A}_h$ on every reduced subbundle P^h (Theorem 7). We can then prove the following theorem.

Theorem 10. *We have the connection A_Σ given by (26) on the fiber bundle $Y \to P/H$ whose restriction $A_h = h^*A_\Sigma$ to the P^h-associated fiber bundle Y^h coincides with the connection $\bar{A}_h$ generated on P^h by the connection A on the principal fiber bundle $P \to X$.*

Proof. Let a principal reduced subbundle $P^h \subset P$ be given, and let $\bar{A}_h$ be the H-connection on P^h (see Theorem 7) generated by the G-connection A on the principal fiber bundle $P \to X$. By Theorem 5, we can extend this connection to the G-connection on P for which h is the integral section of the associated connection

$$\bar{A}_h = dx^\lambda \otimes (\partial_\lambda + A^p_\lambda J^m_p \partial_m)$$

on the P-associated fiber bundle $\Sigma \to X$. With respect to the atlas Ψ^h given by (14) of the fiber bundle P with H-valued transition functions, the Higgs field h takes values in the center of the homogeneous space G/H, and the connection $\bar{A}_h$ is

$$\bar{A}_h = dx^\lambda \otimes (\partial_\lambda + A^a_\lambda e_a). \tag{35}$$

We then obtain

$$A = \bar{A}_h + \Theta = dx^\lambda \otimes (\partial_\lambda + A^a_\lambda e_a) + \Theta^b_\lambda dx^\lambda \otimes e_b, \tag{36}$$

where $\{e_a\}$ is the basis of the right Lie algebra $\mathfrak{h}_r$ and $\{e_b\}$ is the basis of its complement $\mathfrak{f}_r$. Decomposition (36) with respect to an arbitrary atlas of the fiber bundle P has the form

$$A = \bar{A}_h + \Theta, \qquad \Theta = \Theta^p_\lambda dx^\lambda \otimes e_p,$$

and satisfies the relation $\Theta^p_\lambda J^m_p = D^A_\lambda h^m$, where D^A_λ are the covariant derivatives with respect to the associated connection A on the fiber bundle $\Sigma \to X$. We consider the covariant differential

$$D = D^m_\lambda dx^\lambda \otimes \partial_m = (\sigma^m_\lambda - A^p_\lambda J^m_p) dx^\lambda \otimes \partial_m$$

with respect to the associated connection A on $\Sigma \to X$. We can represent this differential as a $V\Sigma$-valued form on the jet manifold $J^1\Sigma$ of the fiber bundle $\Sigma \to X$. Because decomposition (36) holds for any section h of the fiber bundle $\Sigma \to X$, we obtain a $V_G P$-valued form $\Theta = \Theta^p_\lambda dx^\lambda \otimes e_p$ on $J^1\Sigma$, which satisfies the equation

$$\Theta^p_\lambda J^m_p = D^m_\lambda. \tag{37}$$

As a result, we obtain a $V_G P$-valued form

$$A_H = dx^\lambda \otimes (\partial_\lambda + (A^p_\lambda - \Theta^p_\lambda) e_p)$$

on $J^1\Sigma$ whose induction to every submanifold $J^1 h(X) \subset J^1\Sigma$ is the connection $\bar{A}_h$ given by (35) written with respect to the atlas Ψ^h given by (25). Because decomposition (36) holds, Eq. (37) has a solution for any G-connection A. We therefore have the $V_G P$-valued form

$$A_H = dx^\lambda \otimes (\partial_\lambda + (a^p_\lambda - \Theta^p_\lambda) e_p) \tag{38}$$

on the product $J^1\Sigma \times_X J^1 C$ such that for any connection A and for any Higgs field h, the restriction of A_H given by (38) to

$$J^1 h(X) \times A(X) \subset J^1\Sigma \underset{X}{\times} J^1 C$$

is a connection $\bar{A}_h$ given by (35) written with respect to the atlas Ψ^h given by (25). Now let A_Σ in (26) be a connection on the principal H-bundle $P \to \Sigma$. This connection determines the $V_\Sigma Y$-valued form

$$\widetilde{\mathcal{D}} = dx^\lambda \otimes \big(y^i_\lambda - (\mathcal{A}^a_m \sigma^m_\lambda + \mathcal{A}^a_\lambda) I^i_a\big)\partial_i \tag{39}$$

(covariant differential (29)) on configuration space (32). We now assume that for a given connection A on the principal G-bundle $P \to X$, the induced connection $A_h = h^* A_{Y\Sigma}$ given by (28) on Y^h coincides with $\bar{A}_h$ given by (35) for any $h \in \Sigma(X)$. By Proposition 4, we can then define components of form (39) as follows. For a given point

$$(x^\lambda, a^r_\mu, a^r_{\lambda\mu}, \sigma^m, \sigma^m_\lambda, y^i, y^i_\lambda) \in J^1C \underset{X}{\times} J^1Y, \tag{40}$$

let h be a section of the fiber bundle $\Sigma \to X$ whose jet $j^1_x h$ in $x \in X$ is $(\sigma^m, \sigma^m_\lambda)$, i.e.,

$$h^m(x) = \sigma^m, \qquad \partial_\lambda h^m(x) = \sigma^m_\lambda.$$

Let the connection fiber bundle C and the Lie algebra fiber bundle $V_G P$ be endowed with atlases associated with the atlas Ψ^h given by (25). We can then write

$$A_h = \bar{A}_h, \qquad \mathcal{A}^a_m \sigma^m_\lambda + \mathcal{A}^a_\lambda = a^a_\lambda - \Theta^a_\lambda. \tag{41}$$

These equations for the functions $\mathcal{A}^a_m$ and $\mathcal{A}^a_\lambda$ at point (40) have a solution because Θ^a_λ are affine functions of the jet coordinates σ^m_λ.

Having the solution of Eq. (41), we substitute it in the covariant differential $\widetilde{D}$ given by (39) requiring that the matter field Lagrangian be factored in form (34) through the form $\widetilde{D}$ given by (39), called the universal covariant differential determined by the G-connection A on the principal fiber bundle P. As a result, we obtain a gauge theory of gauge potentials of the group G, of matter fields with the symmetry subgroup $H \subset G$, and of classical Higgs fields on configuration space (40).

As mentioned above, an example of a classical Higgs field is the gravitational field of the metric in the gauge theory of gravity on natural fiber bundles with the spontaneous symmetry breaking due to the existence of Dirac spinor fields with the Lorentz group of symmetries or by the geometric equivalence principle [5], [10], [11]. Describing spinor fields in terms of composition bundle (21), we obtain their Lagrangian (34) in the presence of the general linear connection; this Lagrangian is invariant under general covariance transformations [3], [11].

In a more general form, classical Higgs fields were also considered in the theory of spinor fields on the so-called gauge-natural fiber bundles [24].

7. Conclusion

In conclusion, we summarize the main features of the description of spontaneous symmetry breaking in a classical gauge theory on the principal fiber bundle $P \to X$ with the structure group G:

1. The spontaneous symmetry breaking in such a gauge theory is characterized by the reduction of the structure group G of the fiber bundle P to its closed subgroup H.

2. This reduction is ensured by the existence of global sections of the quotient bundle $P/H \to X$ interpreted as classical Higgs fields.

3. Matter fields with the exact symmetry group H in such a gauge theory are described in pairs with the Higgs fields, which are sections of a composition bundle $Y \to P/H \to X$, where $Y \to P/H$ is the fiber bundle associated with $P \to P/H$ and has the structure group H.

4. The fiber bundle $Y \to X$, as was shown, is associated with the initial fiber bundle $P \to X$ and has the structure group G, while the gauge-invariant Lagrangian of matter fields is factored through the vertical covariant differential determined by an H-connection on the principal fiber bundle $P \to P/H$.

5. In the case of Cartan decomposition (18) of the Lie algebra of the group G, this connection can be expressed in terms of the G-connection on the principal fiber bundle $P \to X$, i.e., in terms of the gauge potentials for the group of broken symmetries G.

As a result, we obtain a gauge theory of gauge potentials of the group G, matter fields with the symmetry subgroup $H \subset G$, and classical Higgs fields.

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