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Constraints in Hamiltonian time-dependent mechanics

L. Mangiarotti[a)]
Department of Mathematics and Physics, University of Camerino, 62032 Camerino (MC), Italy

G. Sardanashvily[b)]
Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia

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The key point of the study of constraints in Hamiltonian time-dependent mechanics lies in the fact that a Poisson structure does not provide dynamic equations and a Poisson bracket of constraints with a Hamiltonian is ill-defined. We describe Hamiltonian dynamics in terms of Hamiltonian forms and connections on the vertical cotangent bundle $V^*Q\to\mathbf{R}$ seen as a momentum phase space. A Poisson bracket $\{,\}_V$ on V^*Q is induced by the canonical Poisson bracket $\{,\}_T$ on the cotangent bundle T^*Q. With $\{,\}_V$, an algebra of first and second class time-dependent constraints is described, but we use the pull-back of the evolution equation onto T^*X and the bracket $\{,\}_T$ in order to extend the constraint algorithm to time-dependent constraints. The case of Lagrangian constraints of a degenerate almost regular Lagrangian is studied in detail. One can assign to this Lagrangian L a set of Hamiltonian forms (which are not necessarily degenerate) such that any solution of the corresponding Hamilton equations which lives in the Lagrangian constraint space is a solution of the Lagrange equations for L. In the case of an almost regular quadratic Lagrangian, the complete set of global nondegenerate Hamiltonian forms with the above-mentioned properties is described. We construct the Koszul–Tate resolution of the Lagrangian constraints for this Lagrangian in an explicit form. © *2000 American Institute of Physics.* [S0022-2488(00)03205-9]

I. INTRODUCTION

We study holonomic constraints in Hamiltonian mechanics subject to time-dependent transformations.[1,2] In contrast to the existent formulations of time-dependent mechanics,[3–7] we do not imply any preliminary splitting of its momentum phase space $\Pi=\mathbf{R}\times Z$. From the physical viewpoint, this splitting characterizes a certain reference frame, and is violated by time-dependent transformations, including inertial frame transformations.

Recall that, given such a splitting, Π is endowed with the product of the zero Poisson structure on $\mathbf{R}$ and the Poisson structure on Z. A Hamiltonian $\mathcal{H}$ is defined as a real function on Π. The corresponding Hamiltonian vector field $\vartheta_{\mathcal{H}}$ on Π is vertical with respect to the fibration $\Pi\to\mathbf{R}$. Due to the natural imbedding $\Pi\times_{\mathbf{R}}T\mathbf{R}\to T\Pi$ one introduces the vector field $\gamma_{\mathcal{H}}=\partial_t+\vartheta_{\mathcal{H}}$, where ∂_t is the standard vector field on $\mathbf{R}$. The Hamilton equations are equations for the integral curves of the vector field γ_H, while the evolution equation on the Poisson algebra $C^\infty(\Pi)$ of smooth functions on Π is given by the Lie derivative

$$L_{\gamma_{\mathcal{H}}}f=\partial_t f+\{\mathcal{H},f\}.$$

However, the splitting on the right-hand side of this expression is violated by time-dependent transformations, and a Hamiltonian $\mathcal{H}$ is not scalar under these transformations. Its Poisson bracket with functions $f\in C^\infty(\Pi)$ is ill-defined, and is not maintained under time-dependent

[a)]Electronic mail: mangiaro@camserv.unicam.it
[b)]Electronic mail: sard@grav.phys.msu.su

transformations. This fact is the key point of the study of constraints in Hamiltonian time-dependent mechanics. Therefore, we need something more than a Poisson structure on Π.

A generic momentum phase space of time-dependent mechanics is a fiber bundle $\Pi\to\mathbf{R}$ endowed with a regular Poisson structure whose characteristic distribution belongs to the vertical tangent bundle $V\Pi$ of $\Pi\to\mathbf{R}$.[8] The problem is that this Poisson structure cannot provide dynamic equations. A first-order dynamic equation on $\Pi\to\mathbf{R}$, by definition, is a section of the affine jet bundle $J^1\Pi\to\Pi$, i.e., a connection on $\Pi\to\mathbf{R}$. Being a horizontal vector field, such a connection cannot be a Hamiltonian vector field with respect to the above-mentioned Poisson structure on Π.

Let us consider time-dependent mechanics on a configuration bundle $Q\to\mathbf{R}$. The corresponding momentum phase space is the vertical cotangent bundle $\Pi=V^*Q$, called the Legendre bundle. It is provided with the canonical Poisson structure $\{,\}_V$ such that[9]

$$\zeta^*\{f,g\}_V=\{\zeta^*f,\zeta^*g\}_T, \quad f,g\in C^\infty(V^*Q), \tag{1}$$

where ζ is the natural fibration

$$\zeta:T^*Q\to V^*Q, \tag{2}$$

and $\{,\}_T$ is the nondegenerate Poisson structure on the cotangent bundle T^*Q defined by the canonical symplectic form $d\Xi$ on T^*Q. The characteristic distribution of $\{,\}_V$ coincides with the vertical tangent bundle VV^*Q of $V^*Q\to\mathbf{R}$.

Given a section h of the fiber bundle (2), let us consider the pull-back forms

$$\mathbf{\Theta}=h^*(\Xi\wedge dt), \quad \mathbf{\Omega}=h^*(d\Xi\wedge dt) \tag{3}$$

on V^*Q. It is readily observed that these forms are independent of h, and are canonical on V^*Q. Then a Hamiltonian vector field ϑ_f for a function f on V^*Q is given by the relation

$$\vartheta_f\rfloor\mathbf{\Omega}=-df\wedge dt,$$

while the Poisson bracket (1) is written as

$$\{f,g\}_V dt=\vartheta_g\rfloor\vartheta_f\rfloor\mathbf{\Omega}.$$

Thus, the three-form $\mathbf{\Omega}$ (3) provides V^*Q with the Poisson structure $\{,\}_V$ in an equivalent way, but gives something more as follows.[1,2,10] A connection γ on the Legendre bundle $V^*Q\to\mathbf{R}$ is said to be a Hamiltonian connection if

$$\gamma\rfloor\mathbf{\Omega}=h^*d\Xi=dH,$$

where h is some section of the fiber bundle (2). The form $H=h^*\Xi$ is called a Hamiltonian form. Given a Hamiltonian form H and the associated Hamiltonian connection γ_H, the kernel of the corresponding covariant differential D_{γ_H} provides the Hamilton equations on the Legendre bundle $V^*Q\to\mathbf{R}$, while the Lie derivative

$$d_t f=L_{\gamma_H}f=\gamma_H\rfloor df \tag{4}$$

defines the evolution equation on the Poisson algebra $C^\infty(V^*Q)$.

Remark 1: A generic momentum phase space $\Pi\to\mathbf{R}$ of time-dependent mechanics can be seen locally as the Poisson product over $\mathbf{R}$ of a Legendre bundle $V^*Q\to\mathbf{R}$ and some fiber bundle over $\mathbf{R}$, equipped with the zero Poisson structure.

With the Poisson bracket $\{,\}_V$, the conventional notion of first and second class constraints can be extended to constraints in Hamiltonian time-dependent mechanics, and the classical BRST technique[11,12] can be applied to these constraints. At the same time, since γ_H is not a vertical vector field, the right-hand side of the evolution equation (4) is not expressed in the Poisson bracket unless a reference frame is given. To overcome this difficulty, we consider the pull-back of the equality (4) onto the cotangent bundle T^*Q where its right-hand side takes the form of the Poisson bracket $\{\mathcal{H}^*,\zeta^*f\}_T$ of the pull-back function ζ^*f and the function $\mathcal{H}^*=\partial_t\rfloor(\Xi-\zeta^*H)$ on T^*Q. This Poisson bracket enables us to extend the constraint algorithm of conservative mechanics (and time-dependent mechanics on a product $\mathbf{R}\times Z$[6,7]) to mechanical systems subject to time-dependent transformations. An essential difference between constraints in conservative mechanics and time-dependent mechanics also lies in the fact that Hamiltonian vector fields of first class time-dependent constraints are not generators of gauge symmetries of a Hamiltonian form H. At the same time, we show that gauge symmetries of a Hamiltonian form H generate a coisotropic ideal of first class constraints. Therefore, the BRST technique may be applied to them.

Lagrangian constraints are one of the most important class of constraints studied in quantum theory. If a Lagrangian L of time-dependent mechanics is degenerate, it defines the Lagrangian constraint subspace N_L of the Legendre bundle V^*Q. We show that, for a degenerate almost regular Lagrangian L, there exists at least locally a complete set of weakly associated Hamiltonian forms H such that solutions of the Hamilton equations for H which live in the Lagrangian constraint space N_L exhaust all solutions of the Lagrange equations for L. It is important that, in contrast to associated Hamiltonian forms studied in our previous works,[1,2] these Hamiltonian forms are not necessarily degenerate. Furthermore, we find a complete set of nondegenerate Hamiltonian forms with the above-mentioned properties for a generic almost regular quadratic Lagrangian. We also show that, in this case, the Legendre bundle V^*Q admits the splitting $V^*Q=\mathrm{Ker}\,\sigma\oplus N_L$ over Q, where σ is some morphism. Using the corresponding projection operators, we construct the Koszul–Tate resolution for the Lagrangian constraints N_L of a generic almost regular quadratic Lagrangian L in an explicit form.

The plan of the paper is as follows. Section II presents some technical preliminaries. In Sec. III, we compile the basic facts of Hamiltonian time-dependent mechanics from our previous works. Section IV is devoted to two useful constructions which are the Lagrangian L_H (11) on the jet manifold J^1V^*Q and the above-mentioned bracket $\{\mathcal{H}^*,\zeta^*f\}_T$ (15) on the cotangent bundle T^*Q. We use them for the study of an evolution equation in time-dependent mechanics. The Lagrangian L_H also enables us to follow the standard procedure of Lagrangian formalism in order to describe gauge symmetries in Hamiltonian mechanics. In Sec. V, an ideal of time-dependent constraints is described in algebraic terms. In Sec. VI, we extend our analysis of degenerate Lagrangian and Hamiltonian systems in the previous works[1,2] to weakly associated Hamiltonian forms, which are not necessarily degenerate. Section VII provides the detailed exposition of the case of an almost regular quadratic Lagrangian, appropriate for application to many physical models. One of the results is the existence of a complete set of nondegenerate Hamiltonian forms weakly associated with this Lagrangian; that may be important for quantization. Another one is the splittings (45a) and (46a) of the velocity and momentum phase spaces. Based on these splittings, we obtain the Koszul–Tate resolution for the Lagrangian constraints of an almost regular quadratic Lagrangian. These constraints are reducible in general. Section VIII is devoted to the geometric description of the corresponding antighost fields. In Sec. IX, the above-mentioned Koszul–Tate resolution and the corresponding BRST charge are constructed.

II. TECHNICAL PRELIMINARIES

The following peculiarities of fiber bundles over $\mathbf{R}$ should be emphasized.[2] Their base $\mathbf{R}$ is parametrized by the Cartesian coordinates t with the transition functions $t'=t+\mathrm{const}$, and is provided with the standard vector field ∂_t and the standard one-form dt. A vector field u on a fiber bundle $Y\to\mathbf{R}$ is said to be projectable if $u\rfloor dt$ is constant. From now on, by vector fields on fiber bundles over $\mathbf{R}$ are meant only projectable vector fields.

Let $Y\to\mathbf{R}$ be a fiber bundle coordinated by (t,y^A) and J^1Y its first-order jet manifold, equipped with the adapted coordinates (t,y^A,y^A_t). There is the canonical imbedding

$$\lambda=\partial_t+y^A_t\partial_A:J^1Y\underset{Y}{\hookrightarrow}TY$$

whose image is the affine subbundle of elements $v\in TY$ such that $v\rfloor dt=1$. This subbundle is modeled over the vertical tangent bundle $VY\to Y$. As a consequence, there is one-to-one correspondence between the connections on the fiber bundle $Y\to\mathbf{R}$ and the vector fields Γ on Y such that $\Gamma\rfloor dt=1$. The corresponding covariant differential reads

$$D_\Gamma=\lambda-\Gamma:J^1Y\underset{Y}{\to}VY,\quad \dot y^A\circ D_\Gamma=y_t^A-\Gamma^A.$$

A connection Γ on $Y\to\mathbf{R}$ yields a one-dimensional distribution on Y, transversal to the fibration $Y\to\mathbf{R}$. As a consequence, it defines an atlas of local constant trivializations of $Y\to\mathbf{R}$ whose transition functions are independent of t and $\Gamma=\partial_t$. Conversely, every atlas of local constant trivializations of a fiber bundle $Y\to\mathbf{R}$ sets a connection on $Y\to\mathbf{R}$ which is ∂_t relative to this atlas. In particular, every trivialization of $Y\to\mathbf{R}$ yields a complete connection Γ on Y, and *vice versa*.

Recall the total derivative $d_t=\partial_t+y_t^A\partial_A+\cdots$ and the exterior algebra homomorphism

$$h_0:\phi\, dt+\phi_A\, dy^A\mapsto(\phi+\phi_A y_t^A)dt,$$

which sends exterior forms on $Y\to\mathbf{R}$ onto the horizontal forms on $J^1Y\to\mathbf{R}$.

III. HAMILTONIAN TIME-DEPENDENT DYNAMICS

In this section, we compile some basic facts of Hamiltonian time-dependent mechanics.[1,2,10] Let the momentum phase space of time-dependent mechanics

$$V^*Q \xrightarrow{\pi_Q} Q\xrightarrow{\pi}\mathbf{R}$$

be provided with holonomic coordinates (t,q^i,p_i). These coordinates are canonical for the Poisson structure (1) on V^*Q such that

$$\begin{gathered}\mathbf{\Omega}=dp_i\wedge dq^i\wedge dt,\\ \{f,g\}_V=\partial^i f\partial_i g-\partial^i g\partial_i f,\quad f,g\in C^\infty(V^*Q).\end{gathered}\tag{5}$$

Lemma 1:[1,2] A vector field u on V^*Q is canonical for the Poisson structure $\{,\}_V$ iff the form $u\rfloor\mathbf{\Omega}$ is closed. The closed form $u\rfloor\mathbf{\Omega}$ is exact.

With respect to the Poisson bracket (5), the Hamiltonian vector field ϑ_f for a function f on the Legendre bundle V^*Q is

$$\vartheta_f=\partial^i f\partial_i-\partial_i f\partial^i .$$

It is vertical. Conversely, one can show that every vertical canonical vector field on the Legendre bundle $V^*Q\to\mathbf{R}$ is locally a Hamiltonian vector field.

Proposition 2: Let a connection γ on the Legendre bundle $V^*Q\to\mathbf{R}$ be a canonical vector field for the Poisson structure $\{,\}_V$. Then $\gamma\rfloor\mathbf{\Omega}=dH$, where H is locally a Hamiltonian form. Conversely, any Hamiltonian form

$$H=h^*\Xi=p_i\, dq^i-\mathcal{H}\, dt\tag{6}$$

on the momentum phase space V^*Q admits a unique Hamiltonian connection

$$\gamma_H=\partial_t+\partial^i\mathcal{H}\partial_i-\partial_i\mathcal{H}\partial^i.\tag{7}$$

Remark 2: A glance at expression (6) shows that, given a trivialization of the configuration bundle $Q\to\mathbf{R}$, the Hamiltonian form H (6) is the well-known integral invariant of Poincaré–Cartan where $\mathcal{H}$ is a Hamiltonian.

Hamiltonian forms constitute an affine space modeled over the vector space of horizontal densities $f\,dt$ on $V^*Q\to\mathbf{R}$, i.e., over $C^\infty(V^*Q)$. Accordingly, Hamiltonian connections γ_H make up an affine space modeled over the vector space of Hamiltonian vector fields.

Remark 3: Any bundle morphism

$$\Phi: V^*Q\to J^1Q\subset TQ, \quad \Phi=\partial_t+\Phi^i\partial_i,$$

called a Hamiltonian map, defines the Hamiltonian form

$$H_\Phi=-\Phi\rfloor\mathbf{\Theta}=p_i\,dy^i-p_i\Phi^i\,dt$$

on V^*Q. Conversely, every Hamiltonian form yields the Hamiltonian map

$$\hat{H}=J^1\pi_Q\circ\gamma_H: V^*Q\to J^1Q, \quad q^i_t\circ\hat{H}=\partial^i\mathcal{H}. \tag{8}$$

Let Γ be a connection on $Q\to\mathbf{R}$. It characterizes a reference frame in nonrelativistic time-dependent mechanics.[1,2,13] Indeed, the vector field Γ sets a tangent vector at each point of Q whose vertical part can be seen as the velocity of an ''observer'' at this point. Accordingly, the atlas of local constant trivializations of $Q\to\mathbf{R}$ associated with a connection Γ and, in particular, every trivialization of $Q\to\mathbf{R}$ can also be regarded as a reference frame. Every connection Γ on $Q\to\mathbf{R}$, by definition, is a section of the affine bundle (2), and defines the frame Hamiltonian form

$$H_\Gamma=\Gamma^*\Xi=p_i\,dq^i-p_i\Gamma^i\,dt.$$

The corresponding Hamiltonian connection is the canonical lift

$$V^*\Gamma=\partial_t+\Gamma^i\partial_i-p_i\partial_j\Gamma^i\partial^j$$

of Γ onto $V^*Q\to\mathbf{R}$. Then any Hamiltonian form H on V^*Q admits the splittings

$$H=H_\Gamma-\widetilde{\mathcal{H}}_\Gamma\,dt, \quad \mathcal{H}=p_i\Gamma^i+\widetilde{\mathcal{H}}_\Gamma, \tag{9}$$

where $\widetilde{\mathcal{H}}_\Gamma$ is the energy function with respect to the reference frame Γ [see (18) below].

Given a Hamiltonian form H (6) and the associated Hamiltonian connection γ_H (7), the kernel of the covariant differential D_{γ_H} defines the Hamilton equations

$$q^i_t=\partial^i\mathcal{H}, \tag{10a}$$

$$p_{ti}=-\partial_i\mathcal{H}. \tag{10b}$$

IV. THE EVOLUTION EQUATION AND SYMMETRY CURRENTS

A Hamiltonian form H (6) is the Poincaré–Cartan form for the Lagrangian

$$L_H=h_0(H)=(p_iq^i_t-\mathcal{H})dt \tag{11}$$

on the jet manifold J^1V^*Q. This Lagrangian is a convenient tool in order to apply the standard Lagrangian technique to Hamiltonian time-dependent mechanics. Given a vector field u on $Q\to\mathbf{R}$ and its lift

$$\widetilde{u}=u^t\partial_t+u^i\partial_i-\partial_iu^jp_j\partial^i$$

onto the Legendre bundle $V^*Q\to\mathbf{R}$, we have

$$\mathbf{L}_{\widetilde{u}}H=\mathbf{L}_{j^1\widetilde{u}}L_H=(-u^t\partial_t\mathcal{H}+p_i\partial_tu^i-u^i\partial_i\mathcal{H}+\partial_ju^ip_i\partial^j\mathcal{H})dt. \tag{12}$$

Applying the first variational formula to (12), we observe that the Hamilton equations (10a) and (10b) for H are exactly the Lagrange equations for L_H.

Furthermore, given a function $f\in C^\infty(V^*Q)$ and its pull-back onto J^1V^*Q, let us consider the bracket

$$(f,L_H)=\delta^i f\,\delta_i L_H-\delta_i f\,\delta^i L_H=\mathbf{L}_{\gamma_H}f-d_t f,$$

where δ^i, δ_i are variational derivatives (in the spirit of the Batalin–Vilkovisky antibracket). Then the equation $(f,L_H)=0$ is the evolution equation

$$d_t f=\mathbf{L}_{\gamma_H}f=\partial_t f+\{\mathcal{H},f\}_V \tag{13}$$

in time-dependent mechanics. Note that, taken separately, the terms on its right-hand side are ill-behaved objects under reference frame transformations. With the splitting (9), the evolution equation (13) is brought into the frame-covariant form

$$\mathbf{L}_{\gamma_H}f=V^*\Gamma\rfloor H+\{\widetilde{\mathcal{H}}_\Gamma,f\}_V,$$

but its right-hand side does not reduce to a Poisson bracket.

The following construction enables us to represent the right-hand side of the evolution equation (13) as a pure Poisson bracket. Given a Hamiltonian form $H=h^*\Xi$, let us consider its pull-back ζ^*H onto the contangent bundle T^*Q. It is readily observed that the difference $\Xi-\zeta^*H$ is a horizontal one-form on $T^*Q\to\mathbf{R}$, while

$$\mathcal{H}^*=\partial_t\rfloor(\Xi-\zeta^*H)=p+\mathcal{H} \tag{14}$$

is a function on T^*Q. Then the relation

$$\zeta^*(\mathbf{L}_{\gamma_H}f)=\{\mathcal{H}^*,\zeta^*f\}_T \tag{15}$$

holds for any function $f\in C^\infty(V^*Q)$. In particular, f is an integral of motion iff its bracket (15) vanishes. Note that $\gamma_H=T\zeta(\vartheta_{\mathcal{H}^*})$ where $\vartheta_{\mathcal{H}^*}$ is the Hamiltonian vector field for the function $\mathcal{H}^*$ (14) with respect to the canonical Poisson structure $\{,\}_T$ on T^*Q.

Relation (12) enables us to obtain the conservation laws in Hamiltonian time-dependent mechanics in accordance with the standard procedure in Lagrangian formalism.[1,2,10,14] The first variational formula applied to the Lagrangian L_H (11) leads to the weak identity $\mathbf{L}_{\tilde{u}}H\approx d_t(u\rfloor H)dt$. If the Lie derivative (12) vanishes, we have the conserved symmetry current

$$J_u=u\rfloor dH=p_i u^i-u^t\mathcal{H}, \tag{16}$$

along u. If u is a vertical vector field, J_u is the Noether current

$$J_u(q)=u\rfloor q=p_i u^i,\quad q=p_i\,dq^i\in V^*Q. \tag{17}$$

If $u=\Gamma$ is a connection,

$$J_\Gamma=p_i\Gamma^i-\mathcal{H}=-\widetilde{\mathcal{H}}_\Gamma \tag{18}$$

is the energy function with respect to the reference frame Γ, taken with the minus sign.[1,2,15] Note that the current J_u (16) is conserved iff its bracket $\{\mathcal{H}^*,\zeta^*J_u\}_T$ (15) vanishes.

Proposition 3: Given a Hamiltonian form H, the symmetry currents (16) make up a subalgebra of the Poisson algebra $C^\infty(V^*Q)$:

$$\{J_u,J_{u'}\}_V=J_{[u,u']}. \tag{19}$$

The proof follows from a direct computation.

Remark 4: It is readily observed that all Noether currents (17) also constitute a subalgebra of the Poisson algebra $C^\infty(V^*Q)$ with respect to the bracket (19).

V. TIME-DEPENDENT CONSTRAINTS

With the Poisson bracket $\{,\}_V$, an algebra of time-dependent constraints can be described similarly to that in conservative Hamiltonian mechanics, but we should use relation (15) in order to extend the constraint algorithm to time-dependent constraints.

Let N be a closed imbedded subbundle $i_N:N\hookrightarrow V^*Q$ of the Legendre bundle $V^*Q\to\mathbf{R}$, treated as a constraint space. Note that N is neither Lagrangian nor symplectic submanifold with respect to the Poisson structure $\{,\}_V$. Let us consider the ideal $I_N\subset C^\infty(V^*Q)$ of functions f on V^*Q which vanish on N, i.e., $i_N^*f=0$. Its elements are said to be constraints. There is the isomorphism

$$C^\infty(V^*Q)/I_N\cong C^\infty(N) \tag{20}$$

of associative commutative algebras. By the normalize $\overline{I_N}$ of the ideal I_N is meant the subset of functions of $C^\infty(V^*Q)$ whose Hamiltonian vector fields restrict to vector fields on N,[12] i.e.,

$$\overline{I_N}=\{f\in C^\infty(V^*Q):\{f,g\}_V\in I_N,\forall g\in I_N\}. \tag{21}$$

It follows from the Jacobi identity that the normalizer (21) is a Poisson subalgebra of $C^\infty(V^*Q)$. Put

$$I'_N=\overline{I_N}\cap I_N. \tag{22}$$

This is also a Poisson subalgebra of $\overline{I_N}$. Its elements are called the first class constraints, while the remaining elements of I_N are the second class constraints. It is readily observed that $I_N^2\subset I'_N$.

Remark 5: Let N be a coisotropic submanifold of V^*Q. Then $I_N\subset\overline{I_N}$ and $I_N=I'_N$, i.e., all constraints are of first class.

Let H be a Hamiltonian form on the momentum phase space V^*Q. In accordance with the relation (15), a constraint $f\in I_N$ is preserved with respect to a Hamiltonian form H if the bracket (15) vanishes on the constraint space. It follows that solutions of the Hamilton equations (10a) and (10b) do not leave the constraint space N if

$$\{\mathcal{H}^*,\zeta^*I_N\}_T\subset\zeta^*I_N. \tag{23}$$

If this relation does not hold, let us introduce secondary constraints $\{\mathcal{H}^*,\zeta^*f\}_T$, $f\in I_N$, which belong to $\zeta^*(C^\infty(V^*Q))$. If the set of primary and secondary constraints is not closed with respect to relation (23), one can add the tertiary constraints $\{\mathcal{H}^*,\{\mathcal{H}^*,\zeta^*f_a\}_T\}_T$, and so on.

Let us assume that N is a final constraint space for a Hamiltonian form H. If H satisfies relation (23), so is a Hamiltonian form

$$H_f=H-f\,dt, \tag{24}$$

where $f\in I'_N$ is a first class constraint. Though Hamiltonian forms H and H_f coincide with each other on the constraint space N, the corresponding Hamilton equations have different solutions in N because $dH|_N\neq dH_f|_N$. At the same time, $d(i_N^*H)=d(i_N^*H_f)$. Therefore, let us consider the pull-back, called the constrained Hamiltonian form,

$$H_N=i_N^*H_f, \tag{25}$$

which is the same for all $f\in I'_N$. Note that H_N (25) is not a true Hamiltonian form on $N\to\mathbf{R}$ in general. On sections r of the bundle $N\to\mathbf{R}$, we can write

$$r^*(u_N \rfloor dH_N)=0, \tag{26}$$

where u_N is an arbitrary vertical vector field on $N\to\mathbf{R}$. They are called the constrained Hamilton equations. It is readily observed that, for any Hamiltonian form H_f (24), every solution of the Hamilton equations which lives in the constraint space N is a solution of the constrained Hamilton equations (26).

Let us mention the problem of constructing a generalized Hamiltonian system, similar to that for a Dirac constraint system in conservative mechanics. Let H satisfy the condition $\{\mathcal{H}^*,\zeta^*I'_N\}_T\subset I_N$, whereas $\{\mathcal{H}^*,\zeta^*I_N\}_T\not\subset I_N$. The goal is to find a constraint $f\in I_N$ such that the modified Hamiltonian $H-f\,dt$ would satisfy the condition

$$\{\mathcal{H}^*+\zeta^*f,\zeta^*I_N\}_T\subset\zeta^*I_N .$$

This is an equation for a second-class constraint *f*.

The above construction, except the isomorphism (20), can be applied to any ideal I of $C^\infty(V^*Q)$, treated as an ideal of constraints.[12] In particular, an ideal I is said to be coisotropic if it is a Poisson algebra. In this case, I is a Poisson subalgebra of the normalize $\overline{I}$ (21), and coincides with I' (22).

For instance, since $\zeta^*(\mathbf{L}_{\vartheta_f}H)\neq\{\zeta^*f,\mathcal{H}^*\}_T$, the constraints $f\in I_N$ preserved with respect to a Hamiltonian form H (i.e., $\{\zeta^*f,\mathcal{H}^*\}_T\in I_N$) are not generators of gauge symmetries of H in general. At the same time, the generators of gauge symmetries of a Hamiltonian form H define an ideal of constraints as follows. Let $\mathcal{A}$ be a Lie algebra of generators u of gauge symmetries of a Hamiltonian form H. In accordance with relation (19), the corresponding symmetry currents J_u (16) on V^*Q constitute a Lie algebra with respect to the Poisson bracket on V^*Q. Let $I_{\mathcal{A}}$ denote the ideal of $C^\infty(V^*Q)$ generated by these symmetry currents. It is readily observed that this ideal is coisotropic. Then one can think of $I_{\mathcal{A}}$ as being an ideal of first class constraints compatible with the Hamiltonian form H, i.e.,

$$\{\mathcal{H}^*,\zeta^*I_{\mathcal{A}}\}_T\subset\zeta^*I_{\mathcal{A}} . \tag{27}$$

Note that any Hamiltonian form $H_u=H-J_u\,dt$, $u\in\mathcal{A}$, obeys the same relation (27), but other currents $J_{u'}$ are not conserved with respect to H_u, unless $[u,u']=0$.

Now let $\mathcal{A}$ be an arbitrary Lie algebra of vertical vector fields u on the configuration bundle $Q\to\mathbf{R}$. As was mentioned in Remark 4, the corresponding symmetry currents J_u (17) on V^*Q constitute a Lie algebra and generate the corresponding coisotropic ideal $I_{\mathcal{A}}$ of $C^\infty(V^*Q)$ with respect to the Poisson bracket $\{,\}_V$ on V^*Q.

Proposition 4: Let $\mathcal{A}$ be a finite-dimensional Lie algebra of vertical vector fields on the configuration bundle $Q\to\mathbf{R}$. If there exists a reference frame Γ on $Q\to\mathbf{R}$ such that $[\Gamma,\mathcal{A}]=0$, then there exists a nonframe Hamiltonian form H on the Legendre bundle V^*Q such that $\mathcal{A}$ is the algebra of gauge symmetries of H.

Proof: Let $\overline{\mathcal{A}}$ be the universal enveloping algebra of the Lie algebra of the symmetry currents J_u, $u\in\mathcal{A}$, (17). Then each nonzero element C of its center of order >1 can be written as a polynomial in J_u, and defines the desired Hamiltonian form $H=H_\Gamma-C\,dt$.

VI. LAGRANGIAN CONSTRAINTS

Lagrangian constraints are one of the most important classes of constraints studied in quantum theory. If a Lagrangian of time-dependent mechanics is degenerate, we have the Lagrangian constraint subspace of the Legendre bundle V^*Q and a set of Hamiltonian forms associated with the same Lagrangian.[1,2] Here, we consider weakly associated Hamiltonian forms. In comparison with the above-mentioned associated Hamiltonian forms, a degenerate Lagrangian may admit a nondegenerate weakly associated Hamiltonian form that is essential for quantization.

Remark 6: Let $L=\mathcal{L}\,dt:J^1Q\to\mathbf{R}$ be a Lagrangian on the velocity phase space J^1Q. It yields the Legendre map

$$\hat{L}:J^1Q\underset{Q}{\to}V^*Q,\quad p_i=\pi_i=\partial_i^t\mathcal{L},$$

whose image $N_L=\hat{L}(J^1Q)\subset V^*Q$ is called a Lagrangian constraint space. Besides the Lagrange equations

$$(\partial_i-d_t\partial_i^t)\mathcal{L}=0,\tag{28}$$

we will also refer to the Cartan equations, which can be introduced as follows. Being the Lepagean equivalent of the Lagrangian L on J^1Q [i.e., $L=h_0(H_L)$], the Poincaré–Cartan form

$$H_L=L+\pi_i(dq^i-q_t^i\,dt)\tag{29}$$

is also the Lepagean equivalent of the Lagrangian

$$\overline{L}=\hat{h}_0(H_L)=(\mathcal{L}+(\hat{q}_t^i-q_t^i)\pi_i)dt,\quad \hat{h}_0(dq^i)=\hat{q}_t^i\,dt,\tag{30}$$

on the repeated jet manifold J^1J^1Q, coordinated by $(t,q^i,q_t^i,\hat{q}_t^i,q_{tt}^i)$. The Lagrange equations for $\overline{L}$ are the above-mentioned Cartan equations

$$\partial_i^t\pi_j(\hat{q}_t^j-q_t^j)=0,\quad \partial_i\mathcal{L}-\hat{d}_t\pi_i+(\hat{q}_t^j-q_t^j)\partial_i\pi_j=0.\tag{31}$$

They are equivalent to the Lagrange equations (28) on holonomic sections $\overline{c}=\dot{c}$ of $J^1Q\to\mathbf{R}$ and in the case of regular Lagrangians.

Given a Lagrangian L on the velocity phase space J^1Q, a Hamiltonian form H on the momentum phase space V^*Q is said to be associated with L if H satisfies the relations

$$\hat{L}\circ\hat{H}\circ\hat{L}=\hat{L},\tag{32a}$$

$$H=H_{\hat{H}}+\hat{H}^*L,\tag{32b}$$

where $\hat{H}$ is the Hamiltonian map (8). A glance at relation (32a) shows that $\hat{L}\circ\hat{H}$ is the projector

$$p_i(z)=\pi_i(t,q^i,\partial^j\mathcal{H}(z)),\quad z\in N_L,$$

from V^*Q onto the Lagrangian constraint space N_L. Accordingly, $\hat{H}\circ\hat{L}$ is the projector from J^1Y onto $\hat{H}(N_L)$. A Hamiltonian form is called weakly associated with a Lagrangian L if condition (32b) holds on the Lagrangian constraint space N_L.

Proposition 5:[10,16] If a Hamiltonian map $\Phi:V^*Q\to_Q J^1Q$ obeys relation (32a), then the Hamiltonian form $H=H_\Phi+\Phi^*L$ is weakly associated with the Lagrangian L. If $\Phi=\hat{H}$, then H is associated with L.

The difference between associated and weakly associated Hamiltonian forms lies in the following. Let H be an associated Hamiltonian form, i.e., equality (32b) holds everywhere on V^*Q. It takes the coordinate form

$$\mathcal{H}=p_i\partial^i\mathcal{H}-\mathcal{L}(t,q^j,\partial^j\mathcal{H}).$$

The exterior differential of this equality leads to the relation

$$(p_i-(\partial_i^t\mathcal{L})(t,q^j,\partial_t^j\mathcal{H}))\partial_t^i\partial_t^a\mathcal{H}=0,$$

which shows that an associated Hamiltonian form is degenerate outside the Lagrangian constraint space N_L.

Let us restrict our consideration to almost regular Lagrangians L, i.e., (i) the Lagrangian constraint space N_L is a closed imbedded subbundle $i_N:N_L\to V^*Q$ of the bundle $V^*Q\to Q$, (ii) the Legendre map $\hat{L}:J^1Q\to N_L$ is a fibered manifold, and (iii) the inverse image $\hat{L}^{-1}(z)$ of any point $z\in N_L$ is a connected submanifold of J^1Q.

Proposition 6: A Hamiltonian form H weakly associated with an almost regular Lagrangian L exists iff the fibered manifold $J^1Q\to N_L$ admits a global section.

This fact is an immediate consequence of the above-mentioned conditions (i), (ii) and Proposition 5. Condition (iii) leads to the following property.

Lemma 7:[2,10] The Poincaré–Cartan form H_L for an almost regular Lagrangian L is constant on the connected inverse image $\hat{L}^{-1}(z)$ of any point $z\in N_L$.

Corollary 8: All Hamiltonian forms weakly associated with an almost regular Lagrangian L coincide with each other on the Lagrangian constraint space N_L, and the Poincaré–Cartan form H_L (29) for L is the pull-back

$$H_L=\hat{L}^*H, \quad \pi_i q^i_t-\mathcal{L}=\mathcal{H}(t,q^j,\pi_j), \tag{33}$$

of any such Hamiltonian form H.

It follows that, given Hamiltonian forms H and H' weakly associated with an almost regular Lagrangian L, their difference is $f\,dt$, $f\in I_{N_L}$. However, $\hat{H}|_{N_L}\neq\hat{H}'|_{N_L}$ in general. Therefore, the Hamilton equations for H and H' do not coincide necessarily on the Lagrangian constraint space N_L. Their solutions can leave the Lagrangian constraint space N_L, i.e., relation (23) fails to hold in general.

Theorem 9: Let a section r of $V^*Q\to\mathbf{R}$ be a solution of the Hamilton equations (10a) and (10b) for a Hamiltonian form H weakly associated with an almost regular Lagrangian L. If r lives in the Lagrangian constraint space N_L, the section $c=\pi_Q\circ r$ of $Q\to\mathbf{R}$ satisfies the Lagrange equations (28), while $\overline{c}=\hat{H}\circ r$ obeys the Cartan equations (31).

The proof is based on the relation $\overline{L}=(J^1\hat{L})^*L_H$, where $\overline{L}$ is the Lagrangian (30), while L_H is the Lagrangian (11). This relation is derived from the equality (33). The converse assertion is more intricate.

Theorem 10: Given an almost regular Lagrangian L, let a section $\overline{c}$ of the jet bundle $J^1Q\to\mathbf{R}$ be a solution of the Cartan equations (31). Let H be a Hamiltonian form weakly associated with L, and let H satisfy the relation

$$\hat{H}\circ\hat{L}\circ\overline{c}=\dot{c},$$

where c is the projection of $\overline{c}$ onto Q. Then, the section $r=\hat{L}\circ\overline{c}$ of the Legendre bundle $V^*Q\to\mathbf{R}$ is a solution of the Hamilton equations (10a) and (10b) for H.

We will say that a set of Hamiltonian forms H weakly associated with an almost regular Lagrangian L is complete if, for each solution c of the Lagrange equations, there exists a solution r of the Hamilton equations for a Hamiltonian form H from this set such that $c=\pi_Q\circ r$. By virtue of Theorem 10, a set of weakly associated Hamiltonian forms is complete if, for every solution c on $\mathbf{R}$ of the Lagrange equations for L, there is a Hamiltonian form H from this set which fulfills the relation

$$\hat{H}\circ\hat{L}\circ\dot{c}=\dot{c}. \tag{34}$$

In accordance with Proposition 6, on an open neighborhood in V^*Q of each point $z\in N_L$, there exists a complete set of local Hamiltonian forms weakly associated with an almost regular Lagrangian L.

Given a Hamiltonian form H weakly associated with an almost regular Lagrangian L, let us consider the corresponding constrained Hamiltonian form H_N (25). By virtue of Corollary 8, H_N is the same for all Hamiltonian forms weakly associated with L, and $H_L=\hat{L}^*H_N$. Furthermore, for any Hamiltonian form H weakly associated with an almost regular Lagrangian L, every solution of the Hamilton equations which lives in the Lagrangian constraint space N_L is a solution of the constrained Hamilton equations (26). Using the equality $H_L=\hat{L}^*H_N$, one can show that the constrained Hamilton equations (26) are quasiequivalent to the Cartan equations, i.e., there is a surjection of the set of solutions of the Cartan equations onto the set of solutions of the constrained Hamilton equations.[2,10]

VII. QUADRATIC DEGENERATE SYSTEMS

Let us study the physically relevant case of almost regular quadratic Lagrangians. We show that, in this case, there always exists a complete set of nondegenerate weakly associated Hamiltonian forms.

Given a configuration bundle $Q\to\mathbf{R}$, let us consider a quadratic Lagrangian L which has the coordinate expression

$$\mathcal{L}=\tfrac{1}{2}a_{ij}q_t^iq_t^j+b_iq_t^i+c, \tag{35}$$

where a, b, and c are local functions on Q. This property is coordinate independent due to the affine transformation law of the coordinates q_t^i. The associated Legendre map

$$p_i\circ\hat{L}=a_{ij}q_t^j+b_i \tag{36}$$

is an affine morphism over Q. It defines the corresponding linear morphism

$$\overline{L}:VQ\underset{Q}{\to}V^*Q, \quad p_i\circ\overline{L}=a_{ij}\dot{q}^j. \tag{37}$$

Let the Lagrangian L (35) be almost regular, i.e., the matrix function a_{ij} is of constant rank. Then the Lagrangian constraint space N_L (36) is an affine subbundle of the bundle $V^*Q\to Q$, modeled over the vector subbundle $\overline{N_L}$ (37) of $V^*Q\to Q$. Hence, $N_L\to Q$ has a global section. For the sake of simplicity, let us assume that it is the canonical zero section $\hat{0}(Q)$ of $V^*Q\to Q$. Then $\overline{N}_L=N_L$. Accordingly, the kernel of the Legendre map (36) is an affine subbundle of the affine jet bundle $J^1Q\to Q$, modeled over the kernel of the linear morphism $\overline{L}$ (37). Then there exists a connection

$$\Gamma:Q\to\operatorname{Ker}\hat{L}\subset J^1Q, \quad a_{ij}\Gamma^j+b_i=0, \tag{38}$$

on $Q\to\mathbf{R}$. Connections (38) constitute an affine space modeled over the linear space of vertical vector fields υ on $Q\to\mathbf{R}$, satisfying the conditions

$$a_{ij}\upsilon^j=0 \tag{39}$$

and, as a consequence, the conditions $\upsilon^ib_i=0$.

The matrix a in the Lagrangian L (35) can be seen as a degenerate fiber metric of constant rank in $VQ\to Q$. Then the following corollary of the well-known theorem on a splitting of exact sequences of vector bundles takes place.

Lemma 11: Given a k-dimensional vector bundle $E\to Z$, let a be a section of rank r of the tensor bundle $\overset{2}{\vee}E^*\to Z$. There is a splitting $E=\operatorname{Ker}a\oplus_Z E'$ where $E'=E/\operatorname{Ker}a$ is the quotient bundle, and a is a nondegenerate fiber metric in E'.

Theorem 12: There exists a linear bundle map

$$\sigma:V^*Q\underset{Q}{\to}VQ, \quad \dot{q}^i\circ\sigma=\sigma^{ij}p_j, \tag{40}$$

such that $\overline{L}\circ\sigma\circ i_N=i_N$.

Proof: The map (40) is a solution of the algebraic equations

$$a_{ij}\sigma^{jk}a_{kb}=a_{ib}\,. \tag{41}$$

By virtue of Lemma 11, there exist the bundle slitting

$$VQ=\mathrm{Ker}\,a\underset{Q}{\oplus}E' \tag{42}$$

and an atlas of this bundle such that transition functions of $\mathrm{Ker}\,a$ and E' are independent. Since a is a nondegenerate fiber metric in E', there is an atlas of E' such that a is brought into a diagonal matrix with nonvanishing components a_{AA}. Due to the splitting (42), we have the corresponding bundle splitting

$$V^*Q=(\mathrm{Ker}\,a)^*\underset{Q}{\oplus}\mathrm{Im}\,a. \tag{43}$$

Then the desired map σ is represented by a direct sum $\sigma_1\oplus\sigma_0$ of an arbitrary section σ_1 of the bundle $\overset{2}{\vee}\mathrm{Ker}\,a^*\to Q$ and the section σ_0 of the bundle $\overset{2}{\vee}E'\to Q$, which has nonvanishing components $\sigma^{AA}=(a_{AA})^{-1}$ with respect to the above-mentioned atlas of E'. Moreover, σ satisfies the particular relations

$$\sigma_0=\sigma_0\circ\overline{L}\circ\sigma_0\,,\quad a\circ\sigma_1=0,\quad \sigma_1\circ a=0. \tag{44}$$

Corollary 13: The splitting (42) leads to the splitting

$$J^1Q=\mathcal{S}(J^1Q)\underset{Q}{\oplus}\mathcal{F}(J^1Q)=\mathrm{Ker}\,\hat{L}\underset{Q}{\oplus}\mathrm{Im}(\sigma\circ\hat{L}), \tag{45a}$$

$$q_t^i=\mathcal{S}^i+\mathcal{F}^i=[q_t^i-\sigma_0^{ik}(a_{kj}q_t^j+b_k)]+[\sigma_0^{ik}(a_{kj}q_t^j+b_k)], \tag{45b}$$

while the splitting (43) can be written as

$$V^*Q=\mathcal{R}(V^*Q)\underset{Q}{\oplus}\mathcal{P}(V^*Q)=\mathrm{Ker}\,\sigma_0\underset{Q}{\oplus}N_L\,, \tag{46a}$$

$$p_i=\mathcal{R}_i+\mathcal{P}_i=[p_i-a_{ij}\sigma_0^{jk}p_k]+[a_{ij}\sigma_0^{jk}p_k]. \tag{46b}$$

It is readily observed that, with respect to the coordinates $\mathcal{S}^i$ and $\mathcal{F}^i$ (45b), the Lagrangian (35) reads

$$\mathcal{L}=\tfrac{1}{2}a_{ij}\mathcal{F}^i\mathcal{F}^j+c',$$

while the Lagrangian constraint space is given by the reducible constraints

$$\mathcal{R}_i=p_i-a_{ij}\sigma_0^{jk}p_k=0. \tag{47}$$

Given the linear map σ (40) and the connection Γ (38), let us consider the affine Hamiltonian map

$$\Phi=\hat{\Gamma}+\sigma:V^*Q\to J^1Q,\quad \Phi^i=\Gamma^i+\sigma^{ij}p_j\,, \tag{48}$$

and the Hamiltonian form

$$\begin{aligned} H &= H_\Phi + \Phi^* L \\ &= p_i dq^i - [p_i \Gamma^i + \tfrac{1}{2}\sigma_0^{ij} p_i p_j + \sigma_1^{ij} p_i p_j - c'] dt \\ &= (\mathcal{R}_i + \mathcal{P}_i) dq^i \\ &\quad - [(\mathcal{R}_i + \mathcal{P}_i)\Gamma^i + \tfrac{1}{2}\sigma_0^{ij}\mathcal{P}_i\mathcal{P}_j + \sigma_1^{ij} p_i p_j - c'] dt. \end{aligned} \tag{49}$$

Theorem 14: The Hamiltonian forms (49) parametrized by connections Γ (38) are weakly associated with the Lagrangian (35) and constitute a complete set.

Proof: By the very definitions of Γ and σ, the Hamiltonian map (48) satisfies the condition (32a). Then H is weakly associated with L (35) in accordance with Proposition 5. Let us write the corresponding Hamilton equations (10a) for a section r of the Legendre bundle $V^*Q \to \mathbf{R}$. They are

$$\dot{c} = (\hat{\Gamma} + \sigma) \circ r, \quad c = \pi_Q \circ r. \tag{50}$$

Due to the surjections $\mathcal{S}$ and $\mathcal{F}$ (45a), the Hamilton equations (50) break in two parts

$$\begin{aligned} \mathcal{S}\circ\dot{c} = \Gamma\circ c, &\quad \dot{r}^i - \sigma^{ik}(a_{kj}\dot{r}^j + b_k) = \Gamma^i \circ c, \\ \mathcal{F}\circ\dot{c} = \sigma\circ r, &\quad \sigma^{ik}(a_{kj}\dot{r}^j + b_k) = \sigma^{ik} r_k. \end{aligned} \tag{51}$$

Let c be an arbitrary section of $Q \to \mathbf{R}$, e.g., a solution of the Lagrange equations. There exists a connection Γ (38) such that relation (51) holds, namely, $\Gamma = \mathcal{S}\circ\Gamma'$ where Γ' is a connection on $Q \to \mathbf{R}$ which has c as an integral section. It is easily seen that, in this case, the Hamiltonian map (48) satisfies relation (34) for c. Hence, the Hamiltonian forms (49) constitute a complete set.

It is readily observed that, if $\sigma_1 = 0$, then $\Phi = \hat{H}$ and the Hamiltonian forms (49) are associated with the Lagrangian (35) in accordance with Proposition 5. If σ_1 is nondegenerate, so is the Hamiltonian form (49). Hence, we have different complete sets of Hamiltonian forms (49) for different σ_1. Hamiltonian forms H (49) of such a complete set differ from each other in the term $v^i \mathcal{R}_i$, where v are vertical vector fields (39). If follows from the splitting (46a) that this term vanishes on the Lagrangian constraint space. The corresponding constrained Hamiltonian form $H_N = i_N^* H$ and the constrained Hamilton equations (26) can be written.

VIII. GEOMETRY OF ANTIGHOSTS

We aim to obtain the Koszul–Tate resolution for the constraints (47). Since these constraints are reducible, we need an infinite number of antighost fields in general.[11,12] We follow the terminology of Ref. 12. They are graded by the antighost number r and the Grassmann parity r mod2. Therefore, the following construction generalizes that of simple graded manifolds[17] to commutative graded algebras generated both by odd and even elements. We use an asterisk $(*)$ for the Grassmann parity.

Let $E = E_0 \oplus E_1 \to Z$ be the Whitney sum of vector bundles $E_0 \to Z$ and $E_1 \to Z$ over a paracompact manifold Z. One can think of E as being a bundle of vector superspaces with a typical fiber $V = V_0 \oplus V_1$ where transition functions of E_0 and E_1 are independent. Let us consider the exterior bundle

$$\wedge E^* = \bigoplus_{k=0}^{\infty} \left(\overset{k}{\underset{Z}{\wedge}} E^*\right),$$

which is the tensor bundle $\otimes E^*$ modulo elements

$$e_0 e_0' - e_0' e_0, \quad e_1 e_1' + e_1' e_1, \quad e_0 e_1 - e_1 e_0, \quad e_0, e_0' \in E_{0z}^*, \quad e_1, e_1' \in E_{1z}^*, \quad z \in Z.$$

One can think of $\wedge E^*$ as being the fiber bundle of commutative superalgebras $\wedge V$, which is the tensor product $\vee E_0^* \otimes \wedge E_1^*$ modulo elements

$$e_0 e_1 - e_1 e_0, \quad e_0 \in E_{0z}^*, \quad e_1 \in E_{1z}^*, \quad z \in Z.$$

Global sections of $\wedge E^*$ constitute a commutative graded algebra $\mathcal{A}(Z)$ modeled on the locally free $C^\infty(Z)$-module $E_0^*(Z) \oplus E_1^*(Z)$ of global sections of E^*. This is the product of the commutative algebra $\mathcal{A}_0(Z)$ of global sections of the symmetric bundle $\vee E_0^* \to Z$ and the graded algebra $\mathcal{A}_1(Z)$ of global sections of the exterior bundle $\wedge E_1^* \to Z$.

Remark 7: Let $\mathcal{A}_1$ be the sheaf of sections of the exterior bundle $\wedge E_1^*$. The pair $(Z, \mathcal{A}_1)$ is a graded manifold.[17] By the well-known Batchelor theorem, any graded manifold is isomorphic to a sheaf of sections of some exterior bundle $\wedge F$, but not in a canonical way. If an exterior bundle $\wedge F$ is given, one speaks about a simple graded manifold. Therefore, the construction below can be extended to an arbitrary commutative graded algebra modeled on a locally free $C^\infty(Z)$-module $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ of finite rank. For the sake of brevity, we agree to call $(Z, \mathcal{A})$ a graded manifold, though its generating set contains an even subset $\mathcal{A}_0$. Accordingly, elements of $A(Z)$ are called graded functions.

Let us study the $\mathcal{A}(Z)$-module $\mathrm{Der}\,\mathcal{A}(Z)$ of graded derivations of $\mathcal{A}(Z)$. Recall that by a graded derivation of the commutative graded algebra $\mathcal{A}(Z)$ is meant an endomorphism of $\mathcal{A}(Z)$ such that

$$u(ff') = u(f)f' + (-1)^{[u][f]} f u(f') \tag{52}$$

for the homogeneous elements $u \in \mathrm{Der}\,\mathcal{A}(Z)$ and $f, f' \in \mathcal{A}(Z)$.

Proposition 15: Graded derivations (52) are represented by sections of a vector bundle.

Proof: Let $\{c^a\}$ be the holonomic bases for $E^* \to Z$ with respect to some bundle atlas (z^A, v^i) of $E \to Z$ with transition functions $\{\rho_b^a\}$, i.e., $c'^a = \rho_b^a(z) c^b$. Then graded functions read

$$f = \sum_{k=0} \frac{1}{k!} f_{a_1 \ldots a_k} c^{a_1} \cdots c^{a_k}, \tag{53}$$

where $f_{a_1 \cdots a_k}$ are local functions on Z, and we omit the symbol of an exterior product of elements c. The coordinate transformation law of graded functions (53) is obvious. Due to the canonical splitting $VE = E \times E$, the vertical tangent bundle $VE \to E$ can be provided with the fiber bases $\{\partial_a\}$ dual of $\{c^a\}$. These are fiber bases for $\mathrm{pr}_2 VE = E$. Then any derivation u of $\mathcal{A}(U)$ on a trivialization domain U of E reads

$$u = u^A \partial_A + u^a \partial_a, \tag{54}$$

where u^A, u^a are local graded functions and u acts on $f \in \mathcal{A}(U)$ by the rule

$$u(f_{a_1 \cdots a_k} c^{a_1} \cdots c^{a_k}) = u^A \partial_A (f_{a_1 \cdots a_k}) c^{a_1} \cdots c^{a_k} + u^a f_{a_1 \cdots a_k} \partial_a \rfloor (c^{a_1} \cdots c^{a_k}). \tag{55}$$

This rule implies the corresponding coordinate transformation law

$$u'^A = u^A, \quad u'^a = \rho_j^a u^j + u^A \partial_A (\rho_j^a) c^j \tag{56}$$

of derivations (54). Let us consider the vector bundle $\mathcal{V}_E \to Z$ which is locally isomorphic to the vector bundle

$$\mathcal{V}_E|_U \approx \wedge E^* \underset{Z}{\otimes} (\mathrm{pr}_2 VE \underset{Z}{\oplus} TZ)|_U,$$

and has the transition functions

$$z'^{A}_{i_1\cdots i_k}=\rho^{-1a_1}_{i_1}\cdots\rho^{-1a_k}_{i_k}z^A_{a_1\cdots a_k},$$

$$v'^{i}_{j_1\cdots j_k}=\rho^{-1b_1}_{j_1}\cdots\rho^{-1b_k}_{j_k}\left[\rho^i_j v^j_{b_1\cdots b_k}+\frac{k!}{(k-1)!}z^A_{b_1\cdots b_{k-1}}\partial_A(\rho^i_{b_k})\right]$$

of the bundle coordinates $(z^A_{a_1\cdots a_k},v^i_{b_1\cdots b_k})$, $k=0,...$. These transition functions fulfill the cocycle relations. It is readily observed that, for any trivialization domain U, the $\mathcal{A}$-module Der $\mathcal{A}(U)$ with the transition functions (56) is isomorphic to the $\mathcal{A}$-module of local sections of $\mathcal{V}_E|_U\to U$. One can show that, if $U'\subset U$ are open sets, there is the restriction morphism Der $\mathcal{A}(U)\to$ Der $\mathcal{A}(U')$. It follows that, restricted to an open subset U, every derivation u of $\mathcal{A}(Z)$ coincides with some local section u_U of $\mathcal{V}_E|_U\to U$, whose collection $\{u_U,U\subset Z\}$ defines uniquely a global section of $\mathcal{V}_E\to Z$, called a graded vector field on Z. Graded vector fields constitute a Lie superalgebra with respect to the bracket

$$[u,u']=uu'+(-1)^{[u][u']+1}u'u.$$

Corollary 16: The sheaf of sections of $\mathcal{V}_E\to Z$ is isomorphic to the sheaf of graded derivations of the sheaf $\mathcal{A}$.

There is the exact sequence over Z of vector bundles

$$0\to\wedge E^*\underset{Z}{\otimes}\mathrm{pr}_2VE\to\mathcal{V}_E\to\wedge E^*\underset{Z}{\otimes}TZ\to 0.$$

Its splitting

$$\tilde\gamma:\dot z^A\partial_A\mapsto\dot z^A(\partial_A+\tilde\gamma^a_A\partial_a) \tag{57}$$

transforms every vector field τ on Z into a graded vector field

$$\tau=\tau^A\partial_A\mapsto\nabla_\tau=\tau^A(\partial_A+\tilde\gamma^a_A\partial_a),$$

which is the derivation ∇_τ of $\mathcal{A}(Z)$ such that

$$\nabla_\tau(sf)=(\tau\rfloor ds)f+s\nabla_\tau(f),\quad f\in\mathcal{A}(Z),\quad s\in C^\infty(Z).$$

Thus, one can think of the splitting (57) as being a graded connection on Z. For instance, every linear connection

$$\gamma=dz^A\otimes(\partial_A+\gamma_A{}^a{}_b v^b\partial_a)$$

on the vector bundle $E\to Z$ yields the graded connection

$$\gamma_s=dz^A\otimes(\partial_A+\gamma_A{}^a{}_b c^b\partial_a)$$

on Z such that, for any vector field τ on Z and any graded function f, the graded derivation $\nabla_\tau(f)$ is exactly the covariant derivative of f relative to the connection γ.

The $\wedge E^*$-dual $\mathcal{V}^*_E$ of $\mathcal{V}_E$ is a vector bundle over Z which is locally isomorphic to the vector bundle

$$\mathcal{V}^*_E|_U\approx\wedge E^*\underset{Z}{\otimes}(\mathrm{pr}_2VE^*\underset{Z}{\oplus}T^*Z)|_U.$$

Global sections of this vector bundle constitute the $\mathcal{A}(Z)$-module of exterior graded one-forms $\phi=\phi_A dz^A+\phi_a dc^a$. Then the morphism $\phi:u\to\mathcal{A}(Z)$ can be seen as the interior product

$$u\rfloor\phi=u^A\phi_A+(-1)^{[\phi_a]}u^a\phi_a. \tag{58}$$

Graded k-forms ϕ can be defined as sections of the graded exterior bundle $\wedge^k_Z\mathcal{V}^*_E$ such that

$$\phi\wedge\sigma=(-1)^{|\phi||\sigma|+[\phi][\sigma]}\sigma\wedge\phi,$$

where $|*|$ is the form degree. The interior product (58) is extended to higher graded forms by the rule

$$u\rfloor(\phi\wedge\sigma)=(u\rfloor\phi)\wedge\sigma+(-1)^{|\phi|+[\phi][u]}\phi\wedge(u\rfloor\sigma).$$

The graded exterior differential d of BRST functions is introduced by the condition $u\rfloor df=u(f)$ for an arbitrary BRST vector field u, and is extended uniquely to higher BRST forms by the rules

$$d(\phi\wedge\sigma)=(d\phi)\wedge\sigma+(-1)^{|\phi|}\phi\wedge(d\sigma),\quad d\circ d=0.$$

It takes the coordinate form

$$d\phi=dz^A\wedge\partial_A(\phi)+dc^a\wedge\partial_a(\phi),$$

where the left derivatives ∂_A,∂_a act on the coefficients of graded forms by rule (55), and they are graded commutative with the forms dz^A,dc^a. The Lie derivative of a graded form ϕ along a graded vector field u is given by the familiar formula

$$\mathbf{L}_u\phi=u\rfloor d\phi+d(u\rfloor\phi).$$

IX. THE KOSZUL–TATE RESOLUTION

To construct the vector bundle E of antighosts, let us consider the vertical tangent bundle $V_Q(V^*Q)$ of $V^*Q\to Q$. Let us choose the bundle E as the Whitney sum of the bundles $E_0\oplus E_1$ over V^*Q which are the infinite Whitney sum over V^*Q of the copies of $V_Q(V^*Q)$. We have

$$E=V_Q(V^*Q)\underset{V^*Q}{\oplus}V_Q(V^*Q)\oplus\cdots.$$

This bundle is provided with the holonomic coordinates $(t,q^i,p_i,\dot p_i^{(r)})$, $r=0,1,\ldots$, where $(t,q^i,p_i,\dot p_i^{(2l)})$ are coordinates on E_0, while $(t,q^i,p_i,\dot p_i^{(2l+1)})$ are those on E_1. By r is meant the antighost number. The dual of $E\to V^*Q$ is

$$E^*=V^*_Q(V^*Q)\underset{V^*Q}{\oplus}V^*_Q(V^*Q)\oplus\cdots.$$

It is endowed with the associated fiber bases $\{c_i^{(r)}\}$, $r=1,2,\ldots$, such that $c_i^{(r)}$ have the same linear coordinate transformation law as the coordinates p_i. The corresponding graded vector fields and graded forms are introduced on V^*Q as sections of the vector bundles $\mathcal{V}_E$ and $\mathcal{V}^*_E$, respectively.

The $C^\infty(V^*Q)$-module $\mathcal{A}(V^*Q)$ of graded functions is graded by the antighost number as

$$\mathcal{A}(V^*Q)=\bigoplus_{r=0}^{\infty}\mathcal{N}^\tau,\quad \mathcal{N}^0=C^\infty(V^*Q).$$

Its terms $\mathcal{N}^r$ constitute a complex

$$0\leftarrow C^\infty(V^*Q)\leftarrow\mathcal{N}^1\leftarrow\ldots \tag{59}$$

with respect to the Koszul–Tate differential

$$\begin{aligned}
&\delta: C^\infty(V^*Q) \to 0,\\
&\delta(c_i^{(2l)}) = a_{ij}\sigma_0^{jk} c_k^{(2l-1)}, \quad l>0,\\
&\delta(c_i^{(2l+1)}) = (\delta_i^k - a_{ij}\sigma_0^{jk}) c_k^{(2l)}, \quad l>0,\\
&\delta(c_i^{(1)}) = (\delta_i^k - a_{ij}\sigma_0^{jk}) p_k .
\end{aligned} \tag{60}$$

The nilpotency property $\delta\circ\delta=0$ of this differential is the corollary of relations (41) and (44).

Proposition 17: The complex (59) with respect to the differential (60) is the Koszul–Tate resolution, i.e., its homology groups are

$$H_{k>1}=0, \quad H_0 = C^\infty(V^*Q)/I_{N_L} = C^\infty(N_L).$$

Note that, in different particular cases of the degenerate quadratic Lagrangian (35), the complex (59) may have a subcomplex, which is also the Koszul–Tate resolution. For instance, if the fiber metric a in $VQ\to Q$ is diagonal with respect to a holonomic atlas of VQ, the constraints (47) are irreducible and the complex (59) contains a subcomplex which consists only of the antighosts $c_i^{(1)}$.

Now let us construct the BRST charge $\mathbf{Q}$ such that

$$\delta(f) = \{\mathbf{Q}, f\}, \quad f\in \mathcal{A}(V^*Q),$$

with respect to some Poisson bracket. The problem is to find the Poisson bracket such that $\{f,g\}=0$ for all $f,g\in C^\infty(V^*Q)$.

To overcome this difficulty, one can consider the vertical extension of Hamiltonian formalism onto the configuration bundle $VQ\to\mathbf{R}$.[2,18] The corresponding Legendre bundle $V^*(VQ)$ is isomorphic to $V(V^*Q)$, and is provided with the holonomic coordinates $(t,q^i,p_i,\dot q^i,\dot p_i)$ such that $(q^i,\dot p_i)$ and $(\dot q^i,p_i)$ are conjugate pairs of canonical coordinates. The momentum phase space $V(V^*Q)$ is endowed with the canonical exterior three-form

$$\mathbf{\Omega}_V = \partial_V \mathbf{\Omega} = [d\dot p_i \wedge dq^i + dp_i\wedge d\dot q^i]\wedge dt, \tag{61}$$

where we use the compact notation

$$\dot\partial_i = \frac{\partial}{\partial \dot q^i}, \quad \dot\partial^i = \frac{\partial}{\partial \dot p_i}, \quad \partial_V = \dot q^i\partial_i + \dot p_i \partial^i.$$

The corresponding Poisson bracket on $V(V^*Q)$ reads

$$\{f,g\}_{VV} = \dot\partial^i f\partial_i g + \partial^i f\dot\partial_i g - \partial^i g\dot\partial_i f - \dot\partial^i g\partial_i f.$$

To extend this bracket to graded functions, let us consider the following graded extension of Hamiltonian formalism.[2,19] We will assume that $Q\to\mathbf{R}$ is a vector bundle, and will further denote $\Pi = V^*Q$.

Let us consider the vertical tangent bundle $VV\Pi$. It admits the canonical decomposition

$$VV\Pi = V\Pi \underset{\mathbf{R}}{\oplus} V\Pi \xrightarrow{\mathrm{pr}_1} V\Pi. \tag{62}$$

Let us choose the bundle E as the Whitney sum of the bundles $E_0\oplus E_1$ over $V\Pi$ which are the infinite Whitney sum over $V\Pi$ of the copies of $VV\Pi$. In view of the decomposition (62), we have

$$E = V\Pi \underset{\mathbf{R}}{\oplus} V\Pi \oplus \cdots \xrightarrow{\mathrm{pr}_1} V\Pi.$$

This bundle is provided with the holonomic coordinates $(t, q^i, p_i, \dot q^i_{(r)}, \dot p_i^{(r)})$, $r = 0,1,\ldots$, where $(t, q^i, p_i, \dot q^i_{(2l)}, \dot p_i^{(2l)})$ are coordinates on E_0 and $(t, q^i, p_i, \dot q^i_{(2l+1)}, \dot p_i^{(2l+1)})$ are those on E_1. The dual of $E \to V\Pi$ is

$$E^* = V\Pi \underset{\mathbf{R}}{\oplus} V\Pi^* \oplus \cdots.$$

It is endowed with the associated fiber bases $\{\overline{c}^i_{(r)}, \overline{c}_i^{(r)}, c^i_{(r)}, c_i^{(r)}\}$, $r = 1,\ldots$. The corresponding graded vector fields and graded forms are introduced on $V\Pi$ as sections of the vector bundles $\mathcal{V}_E$ and $\mathcal{V}^*_E$, respectively. Let us complexify these bundles as

$$C \underset{\mathbf{R}}{\otimes} \mathcal{V}_{VV\Pi}, \quad C \underset{\mathbf{R}}{\otimes} \mathcal{V}^*_{VV\Pi}.$$

The BRST extension of the form (61) on V^*Q is the three-form

$$\mathbf{\Omega}_S = \mathbf{\Omega}_V + i \sum_{r=1}^{\infty} (d\overline{c}_i^{(r)} \wedge dc^i_{(r)} - dc_i^{(r)} \wedge d\overline{c}^i_{(r)}) \wedge dt.$$

The corresponding bracket of graded functions on V^*Q reads

$$\{f,g\}_S = \{f,g\}_{VV} - i \sum_{r=1}^{\infty} (-1)^{r[f]} \left[\frac{\partial f}{\partial \overline{c}_i^{(r)}} \frac{\partial g}{\partial c^i_{(r)}} + (-1)^r \frac{\partial f}{\partial \overline{c}^i_{(r)}} \frac{\partial g}{\partial c_i^{(r)}} - \frac{\partial f}{\partial c_i^{(r)}} \frac{\partial g}{\partial \overline{c}^i_{(r)}} - (-1)^r \frac{\partial f}{\partial c^i_{(r)}} \frac{\partial g}{\partial \overline{c}_i^{(r)}} \right]. \tag{63}$$

It satisfies the condition $\{f,g\}_S = -(-1)^{[f][g]}\{g,f\}_S$. Then the desired BRST charge takes the form

$$\mathbf{Q} = i \left[\overline{c}^i_{(1)} (\delta^k_i - a_{ij}\sigma_0^{jk}) p_k + \sum_{l=1}^{\infty} (\overline{c}^i_{(2l)} a_{ij}\sigma_0^{jk} c_k^{(2l-1)} + \overline{c}^i_{(2l+1)} (\delta^k_i - a_{ij}\sigma_0^{jk}) c_k^{(2l)}) \right].$$

Due to the bracket (63), one can use this charge in order to obtain the BRST complex for antighosts $c_i^{(r)}$ and ghosts $\overline{c}^i_{(r)}$ such that

$$\overline{c}^i_{(2l-1)} \mapsto a_{kj}\sigma_0^{ij} \overline{c}^k_{(2l)}, \quad \overline{c}^i_{(2l)} \mapsto -(\delta^i_k - a_{kj}\sigma_0^{ij}) \overline{c}^k_{(2l+1)}, \quad l > 0.$$

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