Covariant spin structure

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Dirac fermion fields associated with different tetrad gravitational fields and under general covariant transformations are described by sections of the composite bundle $S\rightarrow\Sigma\rightarrow X^4$, which is both the Dirac spinor bundle over the tetrad bundle Σ and the natural one over X^4. As a natural bundle, $S\rightarrow X^4$ admits general covariant transformations which are those of Dirac spin structures. A different way is to consider a background spin structure. We find gauge transformations which preserve this spin structure, but act on effective tetrad fields as general covariant transformations.

I. INTRODUCTION

Metric and metric-affine theories of gravity in the absence of fermion fields are formulated on the natural bundles over a world manifold X^4 such that there exist canonical lifts of diffeomorphisms of X^4 onto these bundles. These lifts are general covariant transformations. The invariance of gravitational Lagrangians under general covariant transformations leads to the energy-momentum conservation laws in these gravitation theories.[1–5] A problem arises because of Dirac fermion fields.

Remark: Manifolds throughout are real, finite-dimensional, Hausdorff, second-countable and connected. By a world manifold X^4 is meant a four-dimensional noncompact oriented manifold which is parallelizable. Such a manifold admits a Dirac spin structure which module isomorphisms is unique.[6,7] This property remains true for all spin structures on X^4 which are generated by the twofold universal covering groups.[8]

Recall that a Dirac spin structure on a world manifold X^4 is said to be a pair (P^h,z_h) of a principal spin bundle $P^h\rightarrow X^4$ with the structure spin group $L_s=SL(2,\mathbf{C})$ and a principal bundle morphism

$$z_h:P^h\rightarrow LX, \tag{1}$$

over X^4 from P^h to the principal bundle $LX\rightarrow X^4$ of oriented linear frames in the tangent bundle TX of X^4.[9–11] The structure group of LX is $GL_4=GL^+(4,\mathbf{R})$. Every Dirac spin structure factorizes through the morphism

$$z_h:P^h\rightarrow L^hX\subset LX,$$

where L^hX, called a reduced Lorentz structure, is a principal Lorentz sub-bundle of the frame bundle LX whose structure group is the proper Lorentz group $L=SO^0(1,3)$. Note that a reduced Lorentz structure is not preserved under general covariant transformations of the frame bundle LX. From the physical viewpoint, it means that a Dirac spin structure provides spontaneous breaking of world symmetries.[12,13]

By the well-known theorem,[14] there is one-to-one correspondence between the reduced Lorentz sub-bundles L^hX of the frame bundle LX and the global sections of the quotient bundle

$$\Sigma=LX/L\rightarrow X^4, \tag{2}$$

called the tetrad bundle. Its elements are oriented frames in TX module Lorentz transformations. The bundle Σ is the twofold covering of the bundle Σ_g of pseudo-Euclidean bilinear forms in TX, whose global sections are pseudo-Riemannian metrics on X^4. Global sections h of $\Sigma\rightarrow X^4$ are tetrad fields on X^4. Accordingly, a Dirac spin structure P^h which factorizes through L^hX is said to be associated with the tetrad field h.

Following the standard terminology of gauge theory,[15–18] one may say that tetrad fields, associated with reduced Lorentz structures, are Higgs fields corresponding to spontaneous breaking of world symmetries.[12,13,15]

Dirac fermion fields in the presence of a tetrad field h are described by sections of the P^h-associated spinor bundle

$$S^h=(P^h\times V)/L_s\to X^4, \tag{3}$$

whose typical fiber V carries the spinor representation of the spin group L_s. To describe Dirac fermion fields and, in particular, to be provided with the Dirac operator, the spinor bundle S^h (3) must be represented as a sub-bundle of the bundle of Clifford algebras, that is, as a spinor structure on the cotangent bundle T^*X in the terminology of Lawson and Michelson.[9] The crucial point is that, for different S^h and $S^{h'}$, these representations are not equivalent though the spin structures P^h and $P^{h'}$ are isomorphic (see Sec. IV). Roughly speaking, for different tetrad fields h, the Clifford representations

$$\gamma_h(dx^\lambda)=h^\lambda_a\gamma^a,$$

of co-frames dx^λ by Dirac's matrices are not equivalent.[12,13]

It follows that every Dirac fermion field must be described in a pair (s_h,h) with a certain tetrad field h, and Dirac fermion fields in the presence of different tetrad fields fail to be given by sections of the same spinor bundle. This fact exhibits the physical nature of gravity as a Higgs field.

There are two ways to describe Dirac fermion fields in the presence of different gravitational fields and under general covariant transformations.

(i) Gravitational fields are identified with different tetrad fields on a world manifold X^4, and the totality of fermion-gravitation pairs (s_h,h) is examined. The goal is to construct a bundle over X^4 whose sections exhaust all these pairs.

(ii) A background spin structure (defined, e.g., by the whole fermion matter of the Universe) and the associated tetrad field on a world manifold X^4 are considered, while different gravitational fields lead to different effective tetrad (or metric) fields, which do not change the background spin structure. The key point is to find gauge transformations over diffeomorphisms of X^4 which both keep the background geometry and act on effective tetrad fields as general covariant transformations.

It should be emphasized that, in both variants, the equations of motion are equivalent at least locally, and so are the equations for a gravitational field if a Lagrangian is independent of a background field.

Following the first variant, let us consider the universal twofold covering group $\widetilde{GL}_4$ of the group GL_4 and the corresponding twofold covering bundle $\widetilde{LX}$ of the frame bundle LX.[9,19–21] The bundle $\widetilde{LX}\to X^4$ inherits the general covariant transformations of the frame bundle LX. However, the spinor representation of the group $\widetilde{GL}_4$ is infinite dimensional. Therefore, the $\widetilde{LX}$-associated spinor bundle describes infinite-dimensional ''world'' spinor fields, but not the Dirac ones. The theory of world spinors has been developed.[22]

In contrast with this world spinor model, our purpose here is to describe the totality of familiar Dirac fermion fields on a world manifold, without appealing to the spinor representation of the general linear group.

We use the fact that the frame bundle LX is the principal bundle $LX\to\Sigma$ over the tetrad bundle Σ (2) with the structure Lorentz group L. Since the diagram

$$\begin{array}{ccc} \widetilde{GL}_4 & \longrightarrow & GL_4 \\ \uparrow & & \uparrow \\ L_s & \xrightarrow{z_L} & L \end{array} \tag{4}$$

commutes (see Sec. V), the bundle $\widetilde{LX}$ is an L_s-principal bundle over the same tetrad bundle $\Sigma = \widetilde{LX}/L_s$. Then the commutative diagram

$$\begin{tikzcd} \widetilde{LX} \arrow[rr, "\tilde{z}"] \arrow[dr] & & LX \arrow[dl] \\ & \Sigma & \end{tikzcd} \tag{5}$$

provides a Dirac spin structure on the tetrad bundle Σ. We will show that, for any tetrad field h, the restriction $h^*\widetilde{LX}$ of the L_s-principal bundle $\widetilde{LX}\to\Sigma$ to $h(X^4)\subset\Sigma$ is isomorphic to the L_s-principal sub-bundle P^h of the bundle $\widetilde{LX}\to X^4$ such that the diagram

$$\begin{tikzcd} \widetilde{LX} \arrow[r, "\tilde{z}"] & LX \\ P^h \arrow[r, "z_h"] \arrow[u] & L^hX \arrow[u] \end{tikzcd} \tag{6}$$

commutes. Then general covariant transformations of the bundle $\widetilde{LX}\to X^4$ take the form of automorphisms of the principal spin bundle $\widetilde{LX}\to\Sigma$ over general covariant transformations of the tetrad bundle Σ. These are desired general covariant transformations of Dirac spin structures (1) as the restrictions to $h(X)\subset\Sigma$ of the diagram (5), called the covariant spin structure.

Now let us consider the spinor bundle

$$S=(\widetilde{LX}\times V)/L_s\to\Sigma, \tag{7}$$

associated with the principal spin bundle $\widetilde{LX}\to\Sigma$. Its typical fiber is the Dirac spinor space V. Given a tetrad field h, the restriction h^*S of $S\to\Sigma$ to $h(X)\subset\Sigma$ is a sub-bundle of the composite bundle

$$S\to\Sigma\to X^4. \tag{8}$$

This sub-bundle is isomorphic to the P^h-associated spinor bundle S^h (3) whose sections s_h describe Dirac fermion fields in the presence of the tetrad field h (see Sec. II). It follows that sections of the composite bundle (8) projected onto different tetrad fields $h:X^4\to\Sigma$ exhaust the totality of pairs (s_h,h) of Dirac fermion fields and tetrad fields. The configuration space of this totality is the first order jet manifold J^1S of the composite bundle $S\to X^4$.[5,23] In this model, tetrad fields are dynamic. They are treated as gravitational fields. We will construct the total Dirac operator and the total Lagrangian on the configuration space J^1S whose restrictions to $h(X)\subset\Sigma$, for any tetrad field h, recover the familiar Dirac operator and the familiar Dirac's Lagrangian of fermion fields in the presence of the tetrad field h and a general linear connection K on X^4.

Note that the bundle $S\to X^4$ is not a spinor bundle. It is provided with the structure of the $\widetilde{LX}$-associated bundle with the structure group $\widetilde{GL}_4$ which acts on the typical fiber $(\widetilde{GL}_4\times V)/L_s$ by the induced representation (see Sec. II). Therefore, general covariant transformations of $\widetilde{LX}$ yield the corresponding automorphisms of the bundle $S\to X^4$, which takes the form of automorphisms of the spinor bundle $S\to\Sigma$ over the general covariant transformations of the tetrad bundle $\Sigma\to X^4$. We will construct the canonical lift onto S of vector fields on X^4 which is the generator of these transformations. Then the energy-momentum conservation law can be derived.[5,23]

Following the variant (ii), we consider a background spin structure P^h associated with a background tetrad field h. In this model, gravitational fields are identified with the sections of the LX-associated group bundle $Q\to X$. The canonical morphism $Q\times\Sigma\to\Sigma$ restricted to $h(X)\subset\Sigma$

defines effective tetrad fields.[24] We will construct automorphisms of LX over diffeomorphisms of X which both preserve the background spin structure $P^h \to LX$ and act on effective tetrad fields as general covariant transformations.

II. REDUCED STRUCTURE

The reduced structure language provides the adequate mathematical formulation of gauge models with broken symmetries.[15–18]

Let $\pi_{PX}: P \to X$ be a principal bundle with a structure group G, which acts freely and transitively

$$R_g: p \mapsto pg, \quad p \in P, \quad g \in G, \tag{9}$$

on P on the right. Let

$$Y = (P \times V)/G, \tag{10}$$

be a P-associated bundle with a typical fiber V on which the structure group G acts on the left. Recall that the quotient (10) is defined by identification of elements (p,v) and $(pg, g^{-1}v)$ for all $g \in G$. Its elements will be denoted by $(p,v)\cdot G$. To say more exactly, the bundle Y (10) is canonically associated with P. In particular, every automorphism Φ of a principal bundle P [which, by definition, is equivariant $R_g \circ \Phi = \Phi \circ R_g$, $\forall g \in G$, under the canonical action (9)] yields the corresponding automorphism

$$\Phi_Y: (P \times V)/G \to (\Phi(P) \times V)/G,$$

of the P-associated bundle Y (10). Recall that every vertical automorphism Φ of P takes the form $p \to p\phi(p)$ where ϕ is a G-valued equivariant function on P, i.e., $\phi(pg) = g^{-1}\phi(p)g$, $\forall g \in G$.

Let H be a Lie subgroup of G. We have the composite bundle

$$P \to P/H \to X,$$

where $\pi_{\Sigma X}: P/H \to X$ is a P-associated bundle, denoted by Σ, with the typical fiber G/H, and $\pi_{P\Sigma}: P \to P/H$ is a principal bundle with the structure group H. A H-principal sub-bundle P^h of P is called a reduced structure.[25,26] By the well-known theorem,[14] there is one-to-one correspondence between the global sections h of the quotient bundle $\Sigma \to X$ and the H-principal sub-bundles P^h of P. Such a sub-bundle P^h is isomorphic to the pull-back $h^*P = \pi_{P\Sigma}^{-1}(h(X))$ over X of the bundle $P \to \Sigma$ by h. The following assertion takes place.[5,26]

Proposition 1: Every vertical automorphism Φ of the principal bundle $P \to X$ sends a reduced sub-bundle P^h onto a reduced sub-bundle $P^{h'}$ which is isomorphic to P^h as a H-principal bundle. Conversely, let two reduced sub-bundles P^h and $P^{h'}$ of a principal bundle P be isomorphic to each other as H-principal bundles. Then every isomorphism $\Phi: P^h \to P^{h'}$ over X can be extended to a vertical automorphism of P.

If the quotient G/H is homeomorphic to an Euclidean space, all H-principal sub-bundles of P are isomorphic to each other as H-principal bundles.[27] This also takes place if P is a trivial bundle.

Given a reduced sub-bundle P^h of a principal bundle P, let

$$Y^h = (P^h \times V)/H, \tag{11}$$

be the P^h-associated bundle with a typical fiber V. If $P^{h'}$ is another reduced sub-bundle of P which is isomorphic to P^h, the bundles Y^h and $Y^{h'}$ are isomorphic, but not canonically isomorphic in general.

Proposition 2: Let P^h be a H-principal sub-bundle of a G-principal bundle P. Let Y^h be the P^h-associated bundle (11) with a typical fiber V. If V carries a representation of the whole group G, the fiber bundle Y^h is canonically isomorphic to the P-associated fiber bundle (10).

Proof: Every element of Y can be represented as $(p,v)\cdot G$, $p \in P^h$. Then the desired isomorphism is

$$Y^h \ni (p,v)\cdot H \mapsto (p,v)\cdot G \in Y.$$

It follows that, given a H-principal sub-bundle P^h of P, any P-associated bundle Y with the structure group G is canonically equipped with a structure of the P^h-associated fiber bundle Y^h with the structure group H. Briefly, we will write

$$Y=(P\times V)/G \cong (P^h \times V)/H = Y^h.$$

However, P^h- and $P^{h'}$-associated bundle structures on Y are not equivalent because, given bundle atlases Ψ^h of P^h and $\Psi^{h'}$ of $P^{h'}$, the union of the associated atlases of Y has necessarily G-valued transition functions between the charts from Ψ^h and $\Psi^{h'}$.

In gauge theory on the principal bundle P, sections h of the quotient bundle Σ are treated as Higgs fields, while sections s_h of the P^h-associated bundle Y^h (11) describe matter fields in the presence of the Higgs fields h. From the physical viewpoint, the structure group G of P is said to be the group of broken symmetries because matter fields carries only a representation of its subgroup H, and a reduced structure $P^h \subset P$ which is not preserved under automorphisms of P.

In general, Y^h is not associated or canonically associated with other H-principal sub-bundles of P. It follows that matter fields can be represented only by pairs with Higgs fields.

To describe the totality of these pairs (s_h, h) for all Higgs fields, let us consider the composite bundle

$$Y \xrightarrow{\pi_{Y\Sigma}} \Sigma \xrightarrow{\pi_{\Sigma X}} X, \tag{12}$$

where $Y \to \Sigma$ is the bundle

$$Y_\Sigma = (P\times V)/H,$$

associated with the H-principal bundle $P\to\Sigma$.[5,28] There is the canonical isomorphism $i_h : Y^h \to h^*Y$ of the P^h-associated bundle Y^h to the sub-bundle of $Y\to X$ which is the restriction

$$h^*Y = (h^*P\times V)/H \cong (P^h\times V)/H = Y^h,$$

of the bundle Y_Σ to $h(X)\subset\Sigma$, i.e.,

$$i_h(Y^h) = \pi^{-1}_{Y\Sigma}(h(X)). \tag{13}$$

Then every global section s_h of Y^h corresponds to the global section $i_h \circ s_h$ of the composite bundle (12). Conversely, every global section s of the composite bundle (12) which projects onto a section $h=\pi_{Y\Sigma}\circ s$ of the bundle $\Sigma\to X$ takes its values into the sub-bundle $i_h(Y^h)\subset Y$ in accordance with the relation (13). Hence, there is one-to-one correspondence between the sections of the bundle Y^h and the sections of the composite bundle (12) which cover h.

Remark: The total space of the composite bundle $Y\to X$ (12) has the structure of the P-associated bundle

$$Y=(P\times(G\times V)/H)/G,$$

where the elements (p,g,v) and $(pab, b^{-1}g, a^{-1}v)$ for all $a\in H$ and $b\in G$ are identified. Its typical fiber is the quotient $(G\times V)/H$ of the product $G\times V$ by identification of the elements (g,v) and $(ag, a^{-1}v)$ for all $a\in H$. The group G act on this typical fiber by the rule

$$G\ni b:(g,v)\cdot H \to (bg,v)\cdot H,$$

which is the induced representation of G by the identic representation of H. In particular, if the typical fiber V of the composite bundle $Y\to X$ admits the action of the group G, these two bundle structures on Y are equivalent.

The feature of the dynamics of field systems on composite bundles consists in the following.[5,28] Let the composite bundle Y (12) be coordinated by (x^λ,σ^m,y^i), where (x^λ,σ^m) are bundle coordinates on $\Sigma\to X$. Its first order jet manifold is provided with the adapted coordinates $(x^\lambda,\sigma^m,y^i,\sigma^m_\lambda,y^i_\lambda)$. Let

$$A=dx^\lambda\otimes(\partial_\lambda+A^i_\lambda\partial_i)+d\sigma^m\otimes(\partial_m+A^i_m\partial_i), \tag{14}$$

be a principal connection on the bundle $Y\to\Sigma$. This connection defines the splitting

$$VY=VY_\Sigma\underset{Y}{\oplus}(Y\underset{\Sigma}{\times}V\Sigma),$$

$$\dot y^i\partial_i+\dot\sigma^m\partial_m=(\dot y^i-A^i_m\dot\sigma^m)\partial_i+\dot\sigma^m(\partial_m+A^i_m\partial_i),$$

of vertical tangent bundles. Using this splitting, one can construct the first-order differential operator

$$\begin{gathered}\widetilde{D}:J^1Y\to T^*X\underset{Y}{\otimes}VY_\Sigma,\\ \widetilde{D}=dx^\lambda\otimes(y^i_\lambda-A^i_\lambda-A^i_m\sigma^m_\lambda)\partial_i,\end{gathered} \tag{15}$$

on the composite bundle Y. The operator (15) possesses the following important property. Given a global section h of Σ, its restriction

$$\begin{gathered}\widetilde{D}_h=\widetilde{D}\circ J^1i_h:J^1Y^h\to T^*X\otimes VY^h,\\ \widetilde{D}_h=dx^\lambda\otimes(y^i_\lambda-A^i_\lambda-A^i_m\partial_\lambda h^m)\partial_i,\end{gathered} \tag{16}$$

to Y^h is exactly the familiar covariant differential relative to the principal connection

$$A_h=dx^\lambda\otimes[\partial_\lambda+(A^i_m\partial_\lambda h^m+A^i_\lambda)\partial_i],$$

on the bundle $Y^h\to X$, which is induced by the principal connection (14) on the bundle $Y\to\Sigma$ by the imbedding i_h.

III. LORENTZ STRUCTURE

An example of a reduced structure is a Lorentz reduced structure in gravitation theories which is deduced from the equivalence principle,[15] and accompanies a Dirac fermion matter.[12,13]

Let $\pi_{LX}:LX\to X^4$ be the frame bundle. Given the holonomic frames $\{\partial_\mu\}$ in the tangent bundle TX, every element $\{H_a\}$ of LX takes the form $H_a=H^\mu_a\partial_\mu$, where H^μ_a is a matrix element of the group GL_4. The frame bundle LX is provided with the bundle coordinates (x^λ,H^μ_a). In these coordinates, the canonical action of the structure group GL_4 on LX reads

$$R_g:H^\mu_a\mapsto H^\mu_b g^b_a,\quad g\in GL_4.$$

As is well-known, the frame bundle LX is equipped with the canonical $\mathbf{R}^4$-valued one-form, which is given by the coordinate expression

$$\theta_{LX}=H^a_\mu dx^\mu\otimes t_a, \tag{17}$$

where $\{t_a\}$ is a fixed basis for $\mathbf{R}^4$ and H^b_μ is the inverse matrix of H^μ_a.

Since a world manifold is parallelizable, the structure group GL_4 of the frame bundle LX is reducible to the Lorentz group L. The corresponding L-principal sub-bundle L^hX is a reduced Lorentz structure. Since LX is trivial, any two Lorentz sub-bundles L^hX and $L^{h'}X$ are isomorphic to each other. By virtue of Proposition 1, there exists a vertical bundle automorphism Φ of LX which sends L^hX onto $L^{h'}X$.

As was mentioned above, there is one-to-one correspondence between the Lorentz sub-bundles L^hX of LX and the global sections h of the tetrad bundle Σ (2) with the typical fiber GL_4/L.

Every tetrad field h defines an associated Lorentz atlas $\Psi^h=\{U_\zeta, z_\zeta^h\}$ of the frame bundle LX where the corresponding local sections z_ζ^h of LX take their values into the Lorentz sub-bundle L^hX. Given a Lorentz atlas Ψ^h, the pull-back

$$z_\zeta^{h*}\theta_{LX}=h^a\otimes t_a=h_\lambda^a dx^\lambda\otimes t_a, \tag{18}$$

of the canonical form θ_{LX} (17) by a local section z_ζ^h is said to be a (local) tetrad form. It determines the tetrad coframes $\{h^a\}$ in the cotangent bundle $T^*X\to X^4$. Their coefficients h_μ^a and the inverse matrix elements $h_a^\mu=H_a^\mu\circ z_\zeta^h$ are called tetrad functions.

Given a Lorentz sub-bundle L^hX, let us consider the associated bundle of Minkowski spaces

$$M^hX=(L^hX\times M)/L, \tag{19}$$

where M is provided with the Minkowski metric η. By virtue of Proposition 2, this bundle is isomorphic to the cotangent bundle T^*X. However, the Minkowski structures M^hX and $M^{h'}X$ on T^*X for different tetrad fields h and h' are not equivalent.

IV. DIRAC SPIN STRUCTURE

Every bundle of Minkowski spaces M^hX (19) over a world manifold is extended to the bundle of Clifford algebras C^hX with the fibers generated by the fibers of M^hX.[10] This bundle C^hX has the structure group Aut($\mathbf{C}_{1,3}$) of inner automorphisms of the Clifford algebra $\mathbf{C}_{1,3}$. In general, C^hX does not contain a spinor sub-bundle because a spinor subspace V (a minimal left ideal) of $\mathbf{C}_{1,3}$ is not stable under inner automorphisms of $\mathbf{C}_{1,3}$. As was shown,[29,30] a spinor sub-bundle of C^hX exists if the transition functions of C^hX can be lifted from Aut($\mathbf{C}_{1,3}$) to the Clifford group $G_{1,3}$. This agrees with the usual condition of existence of a spin structure which holds for X^4. Such a spinor sub-bundle is the bundle S^h (3) associated with the universal twofold covering

$$z_h: P^h\to L^hX, \quad z_h\circ R_g=R_{z_L(g)}, \quad \forall g\in L_s,$$

of L^hX. This is the h-associated Dirac spin structure on a world manifold.

There exists the bundle morphism

$$\gamma_h: T^*X\otimes S^h=(P^h\times(M\otimes V))/L_s\to(P^h\times\gamma(M\otimes V))/L_s=S^h, \tag{20}$$

whereby γ is meant the left action of $M\subset\mathbf{C}_{1,3}$ on $V\subset\mathbf{C}_{1,3}$.[12,13] One can think of (20) as being the representation of covectors to X^4 by the Dirac γ-matrices on elements of the spinor bundle S^h. Relative to an atlas $\{z_\zeta\}$ of P^h and to the associated Lorentz atlas $\{z_h\circ z_\zeta\}$ of LX, the representation (20) reads

$$y^A(\gamma_h(h^a(x)\otimes v))=\gamma^{aA}{}_B y^B(v), \quad v\in S_x^h,$$

where y^A are the corresponding bundle coordinates of S^h, and h^a are the tetrad coframes (18). For brevity, we will write

$$\hat h^a=\gamma_h(h^a)=\gamma^a, \quad \hat d x^\lambda=\gamma_h(dx^\lambda)=h_a^\lambda(x)\gamma^a.$$

Let A_h be a principal connection on S^h, and let

$$D: J^1S^h\to T^*X\underset{S^h}{\otimes}S^h,$$

$$D=(y_\lambda^A-A^{ab}{}_\lambda L_{ab}{}^A{}_B y^B)dx^\lambda\otimes\partial_A$$

where

$$L_{ab}=\tfrac{1}{4}[\gamma_a,\gamma_b], \tag{21}$$

be the corresponding covariant differential. The first-order differential Dirac operator is defined on S^h by the composition

$$\begin{gathered}\Delta_h=\gamma_h\circ D:J^1S^h\to T^*X\otimes S^h\to S^h,\\ y^A\circ\Delta_h=h_a^\lambda\gamma^{aA}{}_B(y_\lambda^B-\tfrac{1}{2}A^{ab}{}_\lambda L_{ab}{}^A{}_By^B).\end{gathered} \tag{22}$$

The h-associated spinor bundle S^h is equipped with the fiber spinor metric

$$a_h(v,v')=\tfrac{1}{2}(v^+\gamma^0v'+v'^+\gamma^0v),\qquad v,v'\in S^h.$$

Using this metric and the Dirac operator (22), one can define Dirac's Lagrangian

$$\begin{aligned}L_h=\Bigg\{&\frac{i}{2}h_q^\lambda\Bigg[y_A^+(\gamma^0\gamma^q)^A{}_B\left(y_\lambda^B-\frac{1}{2}A_\lambda{}^{ab}L_{ab}{}^B{}_Cy^C\right)\\ &-\left(y_{\lambda A}^+-\frac{1}{2}A_\lambda{}^{ab}y_C^+L_{ab}^+\right)(\gamma^0\gamma^q)^A{}_By^B\Bigg]-my_A^+(\gamma^0)^A{}_By^B\Bigg\}\det(h_\mu^a),\end{aligned} \tag{23}$$

on J^1S^h which describes Dirac fermion fields in the presence of a tetrad field h and a principal connection A_h on S_h.

We consider the general case of a principal connection A_h on S_h generated by a general linear connection on a world manifold as follows. Let ω_K be a connection form on LX of a general linear connection

$$K=dx^\lambda\otimes\left(\partial_\lambda+K_\lambda{}^\mu{}_\nu\dot{x}^\nu\frac{\partial}{\partial\dot{x}^\mu}\right), \tag{24}$$

on X^4, called a world connection. By virtue of the well-known theorem,[14] the pull-back $z_h^*\omega_L$ over P^h of the Lorentz part ω_L of ω_K is the connection form of the spin connection

$$K_h=dx^\lambda\otimes[\partial_\lambda+\tfrac{1}{4}(\eta^{kb}h_\mu^a-\eta^{ka}h_\mu^b)(\partial_\lambda h_k^\mu-h_k^\nu K_\lambda{}^\mu{}_\nu)L_{ab}{}^A{}_By^B\partial_A], \tag{25}$$

on S^h, where L_{ab} are the generators (21).[23,31–33]

Remark: There is one-to-one correspondence between the world connections and the sections of the quotient bundle

$$C_K=J^1LX/GL_4, \tag{26}$$

whereby J^1LX implies the first-order jet manifold of the frame bundle $LX\to X^4$. With respect to the holonomic frames in TX, the bundle C_K is coordinated by $(x^\lambda,k_\lambda{}^\nu{}_\alpha)$ so that, for any section K of $C_K\to X^4$,

$$k_\lambda{}^\nu{}_\alpha\circ K=K_\lambda{}^\nu{}_\alpha,$$

are the coefficients of the world connection K (24).

Motivated by the connection (25), one can obtain the canonical lift

$$\tilde{\tau}=\tau^\lambda\partial_\lambda+\tfrac{1}{4}(\eta^{kb}h_\mu^a-\eta^{ka}h_\mu^b)(\tau^\lambda\partial_\lambda h_k^\mu-h_k^\nu\partial_\nu\tau^\mu)L_{ab}{}^A{}_By^B\partial_A, \tag{27}$$

of vector fields τ on X onto the spinor bundle S^h.[5,23] The lift (27) is brought into the form

$$\tilde{\tau}=\tau_{\{\}}-\tfrac{1}{4}(\eta^{kb}h_\mu^a-\eta^{ka}h_\mu^b)h_k^\nu\nabla_\nu\tau^\mu L_{ab}{}^A{}_By^B\partial_A,$$

where $\tau_{\{\}}$ is the horizontal lift of τ by means of the spin Levi-Civita connection for the tetrad field h, and $\nabla_\nu\tau^\mu$ are the covariant derivatives of τ relative to the Levi-Civita connection.[34,35]

The canonical lift (27) fails to be a generator of general covariant transformations because it does not involve transformations of tetrad fields. To define general covariant transformations, one must consider spinor structures associated with different tetrad fields. The difficulty arises because, though the principal spinor bundles P^h and $P^{h'}$ are isomorphic, the associated structures of the bundles of Minkowski spaces M^hX and $M^{h'}X$ (19) on the cotangent bundle T^*X are not equivalent, and so are the representations γ_h and $\gamma_{h'}$ (20).[12,13] Indeed, let

$$t^* = t_\mu dx^\mu = t_a h^a = t'_a h'^a,$$

be an element of T^*X. Its representations γ_h and $\gamma_{h'}$ (20) read

$$\gamma_h(t^*) = t_a \gamma^a = t_\mu h^\mu_a \gamma^a, \quad \gamma_{h'}(t^*) = t'_a \gamma^a = t_\mu h'^\mu_a \gamma^a.$$

There is no isomorphism Φ_s of S^h onto $S^{h'}$ which can obey the condition

$$\gamma_{h'}(t^*) = \Phi_s \gamma_h(t^*) \Phi_s^{-1}, \quad \forall t^* \in T^*X.$$

We thus observe the phenomenon of symmetry breaking in gravitation theory which exhibits the physical nature of gravity as a Higgs field.

V. COVARIANT SPIN STRUCTURE

We start from the following two facts.

Remark: The L-principal bundle

$$P_L := GL_4 \to GL_4/L, \tag{28}$$

is trivial. In accordance with the classification theorem,[27] a G-principal bundle over an n-dimensional sphere S^n is trivial if the homotopy group $\pi_{n-1}(G)$ is trivial. The base GL_4/L is homeomorphic to $S^3 \times \mathbf{R}^7$. Let us consider the morphism f_1 of S^3 into GL_4/L, $f_1(p) = (p,0)$, and the pull-back L-principal bundle $f_1^* P_L \to S^3$. Since L is homeomorphic to $RP^3 \times \mathbf{R}^3$ and $\pi_2(L) = 0$, this bundle is trivial. Let f_2 be the projection of GL_4/L onto S^3. Then the pull-back L-principal bundle $f_2^*(f_1^* P_L) \to GL_4/L$ is also trivial. Since the composition $f_1 \circ f_2$ of GL_4/L into GL_4/L is homotopic to the identity morphism of GL_4/L, the bundle $f_2^*(f_1^* P_L) \to GL_4/L$ is equivalent to the bundle P_L.[27] It follows that the bundle (28) is also trivial.

Remark: The diagram (4) commutes. The restriction of the universal covering group $\widetilde{GL}_4 \to GL_4$ to the Lorentz group L is obviously a covering space of L. Let us show that this is the universal covering space. Indeed, any noncontractible cycle in GL_4 belongs to some subgroup $SO(3) \subset GL_4$ and the restriction of the covering bundle $\widetilde{GL}_4 \to GL_4$ to $SO(3)$ is the universal covering of $SO(3)$. Since the L is homotopic to its maximal compact subgroup $SO(3)$, its universal covering space belongs to $\widetilde{GL}_4$.

As a consequence, we have the commutative diagram (6)[36] and the covariant spin structure (5).

The covariant spin structure is unique. Since the bundle $\Sigma \to X^4$ is trivial, the set of non-equivalent spin structures on Σ is in bijective correspondence with the cohomology group $H^1(S^3 \times \mathbf{R}^7 \times X^4; \mathbf{Z}_2)$. Since the cohomology group $H^1(S^3; \mathbf{Z}_2)$ is trivial and a spin structure on S^3 is unique,[37] one can show that nonequivalent spin structures on Σ are in bijective correspondence with those on X^4.

Let us consider the composite spinor bundle S (8) where $S \to \Sigma$ is the spinor bundle (7) associated with the L_s-principal bundle $\widetilde{LX} \to \Sigma$. Given a tetrad field h, there is the canonical isomorphism

$$i_h : S^h = (P^h \times V)/L_s \to (h^* \widetilde{LX} \times V)/L_s,$$

of the h-associated spinor bundle S^h (3) onto the restriction h^*S of the spinor bundle $S \to \Sigma$ to $h(X) \subset \Sigma$. Thence, every global section s_h of the spinor bundle S^h corresponds to the global section $i_h \circ s_h$ of the composite spinor bundle (3). Conversely, every global section s of the composite spinor bundle (8), which projects onto a tetrad field h, takes its values into the sub-bundle $i_h(S^h) \subset S$.

Let the frame bundle $LX \to X^4$ be provided with a holonomic atlas $\{U_\zeta, T\phi_\zeta\}$, and let the principal bundles $\widetilde{LX} \to \Sigma$ and $LX \to \Sigma$ have the associated atlases $\{U_\epsilon, z^s_\epsilon\}$ and $\{U_\epsilon, z_\epsilon = \widetilde{z} \circ z^s_\epsilon\}$, respectively. With these atlases, the composite spinor bundle S is equipped with the bundle coordinates $(x^\lambda, \sigma^\mu_a, y^A)$, where $(x^\lambda, \sigma^\mu_a)$ are coordinates on Σ such that σ^μ_a are the matrix components of the group element $(T\phi_\zeta \circ z_\epsilon)(\sigma)$, $\sigma \in U_\epsilon$, $\pi_{\Sigma X}(\sigma) \in U_\zeta$. For any tetrad field h, we have $(\sigma^\lambda_a \circ h)(x) = h^\lambda_a(x)$ where $h^\lambda_a(x) = H^\lambda_a \circ z_\epsilon \circ h$ are the tetrad functions with respect to the Lorentz atlas $\{z_\epsilon \circ h\}$ of $L^h X$.

The spinor bundle $S \to \Sigma$ is the sub-bundle of the bundle of Clifford algebras which is generated by the bundle of Minkowski spaces

$$E_M = (LX \times M)/L \to \Sigma,$$

associated with the L-principal bundle $LX \to \Sigma$. Since the bundles LX and P_L (28) are trivial, so is the bundle $E_M \to \Sigma$. Hence, it is isomorphic to the product $\Sigma \underset{X}{\times} T^*X$. Then there exists the representation

$$\gamma_\Sigma : T^*X \underset{\Sigma}{\otimes} S = (\widetilde{LX} \times (M \otimes V))/L_s \to (\widetilde{LX} \times \gamma(M \otimes V))/L_s = S, \tag{29}$$

given by the coordinate expression

$$\hat{d}x^\lambda = \gamma_\Sigma(dx^\lambda) = \sigma^\lambda_a \gamma^a.$$

Restricted to $h(X) \subset \Sigma_T$, this representation recovers the morphism γ_h (20).

Using this representation, one can construct the total Dirac operator on the composite spinor bundle S as follows. Since the bundles $\widetilde{LX} \to \Sigma$ and $\Sigma \to X^4$ are trivial, let us consider a principal connection A (14) on the L_s-principal bundle $\widetilde{LX} \to \Sigma$ given by the local connection form

$$A = (A_\lambda{}^{ab} dx^\lambda + A^{kab}_\mu d\sigma^\mu_k) \otimes L_{ab}, \tag{30}$$

$$\begin{gathered} A_\lambda{}^{ab} = -\tfrac{1}{2}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)\sigma^\nu_k K_\lambda{}^\mu{}_\nu, \\ A^{kab}_\mu = \tfrac{1}{2}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu), \end{gathered} \tag{31}$$

where K is a world connection on X^4. This connection defines the associated spin connection

$$A_S = dx^\lambda \otimes (\partial_\lambda + \tfrac{1}{2}A_\lambda{}^{ab} L_{ab}{}^A{}_B y^B \partial_A) + d\sigma^\mu_k \otimes (\partial^k_\mu + \tfrac{1}{2}A^{kab}_\mu L_{ab}{}^A{}_B y^B \partial_A), \tag{32}$$

on the spinor bundle $S \to \Sigma$. The choice of the connection (30) is motivated by the fact that, given a tetrad field h, the restriction of the spin connection (32) to S^h is exactly the spin connection (25).

The connection (32) yields the first order differential operator $\widetilde{D}$ (15) on the composite spinor bundle $S \to X^4$ which reads

$$\widetilde{D}: J^1 S \to T^*X \underset{\Sigma}{\otimes} S,$$

$$\begin{aligned} \widetilde{D} &= dx^\lambda \otimes [y^A_\lambda - \tfrac{1}{2}(A_\lambda{}^{ab} + A^{kab}_\mu \sigma^\mu_{\lambda k}) L_{ab}{}^A{}_B y^B] \partial_A \\ &= dx^\lambda \otimes [y^A_\lambda - \tfrac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k} - \sigma^\nu_k K_\lambda{}^\mu{}_\nu) L_{ab}{}^A{}_B y^B] \partial_A. \end{aligned} \tag{33}$$

The corresponding restriction $\widetilde{D}_h$ (16) of the operator $\widetilde{D}$ (33) to $J^1 S^h \subset J^1 S$ recovers the familiar covariant differential on the h-associated spinor bundle $S^h \to X^4$ relative to the spin connection (27).

Combining (29) and (33), we obtain the first-order differential operator

$$\Delta=\gamma_\Sigma\circ\widetilde{D}:J^1S\to T^*X\underset{\Sigma}{\otimes}S\to S,$$

$$y^B\circ\Delta=\sigma^\lambda_a\gamma^{aB}{}_A[y^A_\lambda-\tfrac{1}{4}(\eta^{kb}\sigma^a_\mu-\eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k}-\sigma^\nu_k K_\lambda{}^\mu{}_\nu)L_{ab}{}^A{}_B y^B],\tag{34}$$

on the composite spinor bundle $S\to X^4$. One can think of Δ as being the total Dirac operator on S since, for every tetrad field h, the restriction of Δ to $J^1S^h\subset J^1S$ is exactly the Dirac operator Δ_h (22) on the spinor bundle S^h in the presence of the tetrad field h and the spin connection (25).

Thus, we come to the model of metric-affine gravity and Dirac fermion fields. The total configuration space of this model is the jet manifold J^1Y of the bundle product

$$Y=C_K\underset{\Sigma}{\times}S,\tag{35}$$

where C_K is the bundle of world connections (26). It is coordinated by $(x^\mu,\sigma^\mu_a,k_\mu{}^\alpha{}_\beta,y^A)$, and J^1Y is provided with the adapted coordinates

$$(x^\mu,\sigma^\mu_a,k_\mu{}^\alpha{}_\beta,y^A,\sigma^\mu_{\lambda a},k_{\lambda\mu}{}^\alpha{}_\beta,y^A_\lambda).$$

The bundle (35) can be endowed with the spin connection

$$A_Y=dx^\lambda\otimes(\partial_\lambda+\widetilde{A}_\lambda{}^{ab}L_{ab}{}^A{}_B y^B\partial_A)+d\sigma^\mu_k\otimes(\partial^k_\mu+A^{kab}_\mu L_{ab}{}^A{}_B y^B\partial_A),\tag{36}$$

where A^{kab}_μ is given by the expression (31), and

$$\widetilde{A}_\lambda{}^{ab}=-\tfrac{1}{2}(\eta^{kb}\sigma^a_\mu-\eta^{ka}\sigma^b_\mu)\sigma^\nu_k k_\lambda{}^\mu{}_\nu.$$

Using the connection (36), we obtain the first-order differential operator

$$\widetilde{D}_Y:J^1Y\to T^*X\underset{\Sigma}{\otimes}S,$$

$$\widetilde{D}_Y=dx^\lambda\otimes[y^A_\lambda-\tfrac{1}{4}(\eta^{kb}\sigma^a_\mu-\eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k}-\sigma^\nu_k k_\lambda{}^\mu{}_\nu)L_{ab}{}^A{}_B y^B]\partial_A,\tag{37}$$

and the total Dirac operator

$$\Delta_Y=\gamma_\Sigma\circ\widetilde{D}:J^1Y\to T^*X\underset{\Sigma}{\otimes}S\to S,$$

$$y^B\circ\Delta_Y=\sigma^\lambda_a\gamma^{aB}{}_A[y^A_\lambda-\tfrac{1}{4}(\eta^{kb}\sigma^a_\mu-\eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k}-\sigma^\nu_k k_\lambda{}^\mu{}_\nu)L_{ab}{}^A{}_B y^B],\tag{38}$$

on the bundle $Y\to X^4$. Given a section $K:X\to C_K$, the restrictions of the spin connection A_Y (36), the operator $\widetilde{D}_Y$ (37) and the Dirac operator Δ_Y (38) to K^*Y are exactly the spin connection (32) and the operators (33) and (34), respectively.

The total Lagrangian on the configuration space J^1Y of metric-affine gravity and fermion fields is the sum

$$L=L_{MA}+L_D.\tag{39}$$

A metric-affine Lagrangian L_{MA} depends on the metric coordinates

$$\sigma_{\mu\nu}=\sigma^a_\mu\sigma^b_\nu\eta_{ab}\tag{40}$$

and the curvature

$$R_{\lambda\mu}{}^\alpha{}_\beta=k_{\lambda\mu}{}^\alpha{}_\beta-k_{\mu\lambda}{}^\alpha{}_\beta+k_\mu{}^\alpha{}_\epsilon k_\lambda{}^\epsilon{}_\beta-k_\lambda{}^\alpha{}_\epsilon k_\mu{}^\epsilon{}_\beta.$$

Dirac's Lagrangian is

$$L_D=\left\{\frac{i}{2}\sigma^\lambda_q\left[y^+_A(\gamma^0\gamma^q)^A{}_B\left(y^B_\lambda-\frac{1}{4}(\eta^{kb}\sigma^a_\mu-\eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k}-\sigma^\nu_k k_\lambda{}^\mu{}_\nu)L_{ab}{}^B{}_C y^C\right)\right.\right.$$
$$\left.-\left(y^+_{\lambda A}-\frac{1}{4}(\eta^{kb}\sigma^a_\mu-\eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k}-\sigma^\nu_k k_\lambda{}^\mu{}_\nu)y^+_C L^+_{ab}{}^C{}_A\right)(\gamma^0\gamma^q)^A{}_B y^B\right]$$
$$\left.-my^+_A(\gamma^0)^A{}_B y^B\right\}\sqrt{|\sigma|},\quad \sigma=\det(\sigma_{\mu\nu}). \tag{41}$$

By construction, it is Hermitian. Note that, in fact, the Lagrangian L_D depends only on the torsion $k_\lambda{}^\mu{}_\nu - k_{\nu\lambda}^\mu$ of a world connection, while the pseudo-Riemannian part is given by the derivative coordinates $\sigma^\mu_{\lambda k}$.

VI. GENERAL COVARIANT TRANSFORMATIONS

The frame bundle $LX\to X^4$ belongs to the category of natural bundles. Every diffeomorphism f of X^4 gives rise canonically to the automorphism

$$\tilde f:(x^\lambda,H^\lambda{}_a)\mapsto(f^\lambda(x),\partial_\mu f^\lambda H^\mu{}_a), \tag{42}$$

of LX and to the corresponding automorphisms (general covariant transformations)

$$\tilde f: T=(LX\times W)/GL_4\to(\tilde f(LX)\times W)/GL_4,$$

of any LX-associated bundle T. In particular, if $T=TX$, the lift $\tilde f = Tf$ is the familiar tangent morphism to f.

The lift (42) yields the canonical horizontal lift $\tilde\tau$ of every vector field τ on X^4 onto LX-associated bundles. For instance, such a lift onto a tensor bundle $(\overset{m}{\otimes}TX)\otimes(\overset{k}{\otimes}T^*X)$ reads

$$\tilde\tau=\tau^\mu\partial_\mu+[\partial_\nu\tau^{\alpha_1}\dot x^{\nu\alpha_2\cdots\alpha_m}_{\beta_1\cdots\beta_k}+\cdots-\partial_{\beta_1}\tau^\nu\dot x^{\alpha_1\cdots\alpha_m}_{\nu\beta_2\cdots\beta_k}-\cdots]\frac{\partial}{\partial\dot x^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_k}}.$$

There also exists the canonical lift

$$\tilde\tau_K=\tau^\mu\partial_\mu+[\partial_\nu\tau^\alpha k_\mu{}^\nu{}_\beta-\partial_\beta\tau^\nu k_\mu{}^\alpha{}_\nu-\partial_\mu\tau^\nu k_\nu{}^\alpha{}_\beta+\partial_{\mu\beta}\tau^\alpha]\frac{2}{\partial k_\mu{}^\alpha{}_\beta}, \tag{43}$$

of τ onto the bundle of world connections C_K (26), though this is not a LX-associated bundle.

Since the covariant spin structure is unique, the $\widetilde{GL}_4$-principal bundle $\widetilde{LX}\to X^4$ as well as the frame bundle LX admits the lift of any diffeomorphism f of the base X^4.[19]

This lift is defined by the commutative diagram

$$\begin{array}{ccc}
\widetilde{LX} & \xrightarrow{\tilde f} & \widetilde{LX}\\
{\scriptstyle\tilde z}\downarrow & & \downarrow{\scriptstyle\tilde z}\\
LX & \xrightarrow{\tilde f} & LX\\
\downarrow & & \downarrow\\
X & \xrightarrow{f} & X
\end{array}$$

The associated morphism of the spinor bundle $S\to\Sigma$ is given by the relation

$$\tilde f_S:(p,v)\cdot L_s\to(\tilde f(p),v)\cdot L_s,\quad p\in\widetilde{LX},\quad v\in S.$$

Because $\widetilde{f}$ is equivarient, this is a fiber-to-fiber automorphism of the bundle $S\to\Sigma$ over the automorphism $\widetilde{f}_\Sigma$ of the tetrad bundle $\Sigma\to X^4$ induced by the diffeomorphism f of X^4. Thus, we have the commutative diagram

$$\begin{tikzcd} S \arrow[r, "\widetilde{f}_S"] \arrow[d] & S \arrow[d] \\ \Sigma \arrow[r, "\widetilde{f}_\Sigma"] \arrow[d] & \Sigma \arrow[d] \\ X^4 \arrow[r, "f"] & X^4 \end{tikzcd}$$

of general covariant transformations of the spinor bundle S.

Accordingly, there exists a canonical lift $\widetilde{\tau}_S$ onto S of every vector field τ on X^4. The goal is to find its coordinate expression. Difficulties arise because the tetrad coordinates σ^μ_a of Σ depend on the choice of an atlas of the bundle $LX\to\Sigma$. Therefore, noncanonical vertical components appear in the coordinate expression of $\widetilde{\tau}_S$.

The lift τ_Σ of a vector field τ on X^4 onto the tetrad bundle Σ can be derived from the relation (40) and the canonical lift

$$\widetilde{\tau}=\tau^\lambda\partial_\lambda+(\partial_\nu\tau^\alpha\sigma^{\nu\beta}+\partial_\nu\tau^\beta\sigma^{\nu\alpha})\frac{\partial}{\partial\sigma^{\alpha\beta}}, \tag{44}$$

of τ onto the bundle of pseudo-Riemannian metric Σ_g identified with an open sub-bundle of the tensor bundle $\overset{2}{\vee}TX$, and provided with coordinates $(x^\lambda,\sigma^{\mu\nu})$. We have

$$\tau_\Sigma=\tau^\lambda\partial_\lambda+\partial_\nu\tau^\mu\sigma^\nu_c\frac{\partial}{\partial\sigma^\mu_c}+Q^\mu_c\frac{\partial}{\partial\sigma^\mu_c},$$

where the terms Q^μ_c obey the condition

$$(Q^\mu_a\sigma^\nu_b+Q^\nu_a\sigma^\mu_b)\eta^{ab}=0.$$

Let us consider a horizontal lift of the vector field τ_Σ onto the spinor bundle $S\to\Sigma$ by means of the spin connection (32). It reads

$$A_S\widetilde{\tau}_\Sigma=\tau^\lambda\partial_\lambda+\partial_\nu\tau^\mu\sigma^\nu_c\frac{\partial}{\partial\sigma^\mu_c}+\frac{1}{4}(\eta^{kb}\sigma^a_\mu-\eta^{ka}\sigma^b_\mu)\sigma^\nu_k(\partial_\nu\tau^\mu-K_\lambda{}^\mu{}_\nu\tau^\nu)(L_{ab}{}^A{}_By^B\partial_A+L^+_{ab}{}^A{}_By^+_A\partial^B)$$
$$+Q^\mu_c\frac{\partial}{\partial\sigma^\mu_c}+\frac{1}{4}Q^\mu_k(\eta^{kb}\sigma^a_\mu-\eta^{ka}\sigma^b_\mu)(L_{ab}{}^A{}_By^B\partial_A+L^+_{ab}{}^A{}_By^+_A\partial^B).$$

This leads us to the desired canonical lift of τ onto S:

$$\widetilde{\tau}_S=\tau^\lambda\partial_\lambda+\partial_\nu\tau^\mu\sigma^\nu_c\frac{\partial}{\partial\sigma^\mu_c}+Q^\mu_c\frac{\partial}{\partial\sigma^\mu_c}+\frac{1}{4}Q^\mu_k(\eta^{kb}\sigma^a_\mu-\eta^{ka}\sigma^b_\mu)(L_{ab}{}^A{}_By^B\partial_A+L^+_{ab}{}^A{}_By^+_A\partial^B),$$

which can be brought into the form

$$\widetilde{\tau}_S=\tau^\lambda\partial_\lambda+\partial_\nu\tau^\mu\sigma^\nu_c\frac{\partial}{\partial\sigma^\mu_c}+\vartheta,$$

$$\vartheta = \tfrac{1}{4}Q^{\mu}_{k}(\eta^{kb}\sigma^{a}_{\mu} - \eta^{ka}\sigma^{b}_{\mu})\left[-L_{ab}{}^{d}{}_{c}\sigma^{\nu}_{d}\frac{\partial}{\partial\sigma^{\nu}_{c}} + L_{ab}{}^{A}{}_{B}y^{B}\partial_{A} + L^{+}_{ab}{}^{A}{}_{B}y^{+}_{A}\partial^{B}\right],$$

where $L_{ab}{}^{d}{}_{c}$ are generators of the Lorentz group in the Minkowski space. The term ϑ is the above-mentioned noncanonical part of this lift. The corresponding total vector field on the fibred product Y (35) reads

$$\widetilde{\tau}_{Y} = \widetilde{\tau} + \vartheta,$$

$$\widetilde{\tau} = \tau^{\lambda}\partial_{\lambda} + \partial_{\nu}\tau^{\mu}\sigma^{\nu}_{c}\frac{\partial}{\partial\sigma^{\mu}_{c}} + \widetilde{\tau}_{K}, \tag{45}$$

where $\overline{\tau}_K$ is the canonical lift (43) onto C_K. Its canonical part $\overline{\tau}$ (45) is the generator of a local one-parameter group of general covariant transformations of the bundle Y, whereas the vertical vector field ϑ is the generator of a local one-parameter group of vertical Lorentz automorphisms of the bundle $S \to \Sigma$. By construction, the total Lagrangian L (39) obeys the relations

$$\mathbf{L}_{J^1\vartheta}L_D = 0, \tag{46}$$

$$\mathbf{L}_{J^1\widetilde{\tau}}L_{MA} = 0, \quad \mathbf{L}_{J^1\widetilde{\tau}}L_D = 0. \tag{47}$$

The relation (46) leads to the Nöther conservation law, while the equalities (47) lead to the energy-momentum one.[23]

VII. BACKGROUND SPIN STRUCTURE

Let us consider the case of a background spin structure and the corresponding background tetrad field h.

Given a tetrad field h, any general covariant transformation of the frame bundle LX can be written as the composition $\widetilde{f} = \Phi \circ \widetilde{f}_h$ of its automorphism $\widetilde{f}_h$ over f which preserves L^hX and some vertical automorphism

$$\Phi : p \mapsto p\,\phi(p), \quad p \in LX, \tag{48}$$

where ϕ is a GL_4-valued equivariant function on LX. Since LX is trivial, the automorphism $\widetilde{f}_h$ exists. Indeed, let z^h be a global section of L^hX. Put

$$\widetilde{f}_h : L_xX \ni p = z^h(x)g \mapsto z^h(f(x))g \in L_{f(x)}X.$$

The automorphism $\widetilde{f}_h$ restricted to L^hX induces an automorphism of the principal bundle P^h and the corresponding automorphism $\widetilde{f}_s$ of the spinor bundle S^h, which preserves the representation (20).

Turn now to the vertical automorphism Φ. Let us consider the group bundle $Q \to X$ associated with LX. Its typical fiber is the group GL_4 which acts on itself by the adjoint representation. Let $(x^\lambda, q^\lambda_\mu)$ be coordinates on Q. There exist the left and right canonical actions of Q on any LX-associated bundle T:

$$\rho_{l,r} : Q \underset{X}{\times} T \to T,$$

$$\rho_l : ((p,g)\cdot GL_4, (p,w)\cdot GL_4) \mapsto (p,gw)\cdot GL_4,$$

$$\rho_r : ((p,g)\cdot GL_4, (p,w)\cdot GL_4) \mapsto (p,g^{-1}w)\cdot GL_4.$$

Given a vertical automorphism Φ (48) of LX, the corresponding vertical automorphisms of an associated bundle T and the group bundle Q read

$$\Phi : (p,w)\cdot GL_4 \mapsto (p,\phi(p)w)\cdot GL_4,$$

$$\Phi:(p,g)\cdot GL_4 \mapsto (p,\phi(p)g\phi^{-1}(p))\cdot GL_4.$$

For any Φ (48), there exists the fiber-to-fiber morphism

$$\overline{\Phi}:(p,q)\cdot GL_4 \mapsto (p,\phi(p)q)\cdot GL_4,$$

of the group bundle Q such that

$$\rho_l(\overline{\Phi}(Q)\times T)=\Phi(\rho_l(Q\times T)), \tag{49}$$

$$\rho_r(\overline{\Phi}(Q)\times\Phi(T))=\rho_r(Q\times T). \tag{50}$$

For instance, if $T=T^*X$, the expressions (49) and (50) take the coordinate form

$$\rho_r:(x^\lambda,q^\lambda_\mu,\dot{x}_\mu)\mapsto(x^\lambda,\dot{x}_\lambda q^\lambda_\mu),$$

$$\overline{\Phi}:(x^\lambda,q^\lambda_\mu)\mapsto(x^\lambda,S^\lambda_\nu q^\nu_\mu),$$

$$\rho_r(x^\lambda,S^\lambda_\nu q^\nu_\mu,\dot{x}_\alpha(S^{-1})^\alpha_\lambda)=(x^\lambda,\dot{x}_\lambda q^\lambda_\mu).$$

Hence, we obtain the representation

$$\gamma_Q:(Q\times T^*X)\underset{Q}{\otimes}(Q\times S^h)\to(Q\times S^h),$$

$$\gamma_Q=\gamma_h\circ\rho_r:(q,t^*)\mapsto\dot{x}_\lambda q^\lambda_\mu\hat{d}x^\mu=\dot{x}_\lambda q^\lambda_\mu h^\mu_a\gamma^a, \tag{51}$$

on elements of the spinor bundle S^h. Let q_e be the canonical global section of the group bundle $Q\to X^4$ whose values are the unit elements of the fibers of Q. Then the representation γ_Q (51) restricted to $q_e(X^4)$ comes to the representation γ_h (20).

Let h be a background tetrad field, while sections $q(x)$ of the group bundle Q are dynamic variables treated as gravitational fields. There is the canonical morphism

$$\rho_l:Q\underset{X}{\times}\Sigma\to\Sigma,$$

$$\rho_l:((p,g)\cdot GL_4,(p,\sigma)\cdot GL_4)\mapsto(p,g\sigma)\cdot GL_4,\quad p\in LX,$$

$$\rho_l:(x^\lambda,q^\lambda_\mu,\sigma^\mu_a)\mapsto(x^\lambda,q^\lambda_\mu\sigma^\mu_a).$$

This morphism restricted to $h(X^4)\subset\Sigma$ takes the form

$$\rho_h:Q\to\Sigma,$$

$$\rho_h:((p,g)\cdot L,(p,\sigma_0)\cdot L)\to(p,g\sigma_0)\cdot L,\quad p\in L^hX,$$

$$\rho_h:(x^\lambda,q^\lambda_\mu)\mapsto(x^\lambda,q^\lambda_\mu h^\mu_a), \tag{52}$$

where σ_0 is the center of the quotient GL_4/L.

Let Σ_h, coordinated by $\overline{\sigma}^\mu_a$, be the quotient of the bundle Q by the kernel $\mathrm{Ker}_h\rho_h$ of the morphism (52) with respect to the section h. This is isomorphic to the tetrad bundle Σ provided with the Lorentz structure of a L^hX-associated bundle. Then the representation (51), which is constant on $\mathrm{Ker}_h\rho_h$, reduces to the representation

$$(\Sigma_h\times T^*X)\underset{\Sigma_h}{\otimes}(\Sigma_h\times S^h)\to(\Sigma_h\times S^h),$$

$$(\widetilde{\sigma},t^*)\mapsto\dot{x}_\lambda\widetilde{\sigma}^\lambda_a\gamma^a.$$

Thence, one can think of a section $\widetilde{h}\neq h$ of the bundle Σ_h as being an effective tetrad field, and can treat $\widetilde{g}^{\mu\nu}=\widetilde{h}^\mu_a\widetilde{h}^\nu_b\eta_{ab}$ as an effective metric. It should be emphasized that the Greek indices go down and go up by means of the background metric $g^{\mu\nu}=h^\mu_a h^\nu_b\eta_{ab}$.

Given a general covariant transformation $\widetilde{f}=\Phi\circ\widetilde{f}_h$ of the frame bundle LX, let us consider the morphism

$$\widetilde{f}_Q:\quad Q\to\overline{\Phi}\circ\widetilde{f}_h(Q),\quad S^h\to\widetilde{f}_s(S^h),\quad T^*X\to\widetilde{f}(T^*X). \tag{53}$$

This preserves the representation (51), i.e., $\gamma_Q\circ\widetilde{f}_Q=\widetilde{f}_s\circ\gamma_Q$, and yields the general covariant transformation $\widetilde{\sigma}^\lambda_a\mapsto\partial_\mu f^\lambda\widetilde{\sigma}^\mu_a$ of the bundle Σ_h.

Using a spin connection K_h (25) and the representation γ_Q (51), one can construct the Dirac operator

$$\Delta_Q=q^\lambda{}_\mu h^\mu_a\gamma^a D_\lambda.$$

Restricted to $q_e(X^4)$, this operator recovers the Dirac operator Δ_h (22) on the spinor bundle S^h in the presence of the background tetrad field h and the world connection K.

In particular, we come to the metric-affine generalization of Logunov's model[38] whose configuration space is the jet manifold J^1Y of the product

$$Y=Q\underset{X}{\times}C_K\underset{X}{\times}S^h. \tag{54}$$

It is coordinated by $(x^\mu,q^\mu_\nu,k_\alpha{}^\mu{}_\nu,y^A)$. A total Lagrangian on this configuration space is the sum

$$L=L_{MA}+L_D+L_q(q,g), \tag{55}$$

where L_{MA} is a metric-affine Lagrangian expressed into the effective metric $\widetilde{\sigma}^{\mu\nu}=\widetilde{\sigma}^\mu_a\widetilde{\sigma}^\nu_b\eta^{ab}$, L_D is Dirac's Lagrangian (41), expressed into the effective tetrad coordinates $\widetilde{\sigma}^\lambda_a=q^\lambda_\mu h^\mu_a$, while the Lagrangian L_q depends on gravitational fields q and the background metric g.

For the sake of simplicity, let us replace the bundle Q in the product (54) with the bundle Σ_h. There exists the canonical lift $\widetilde{\tau}_{Y'}$ onto $\Sigma_h\times C_K\times S^h$ of every vector field τ on X^4. It coincides with the vector field $\widetilde{\tau}_Y$ (45) where σ^λ_a are replaced with $\widetilde{\sigma}^\lambda_a$. This lift is the generator of gauge transformations induced by morphisms $\widetilde{f}_Q$ (53). Then using the standard procedure,[5,23] obtain the energy-momentum conservation law

$$\partial_\lambda(\tau^\lambda L_q)+(\partial_\alpha\tau^\mu\widetilde{g}^{\alpha\nu}+\partial_\alpha\tau^\nu\widetilde{g}^{\alpha\mu})\,\frac{\partial L_q}{\partial\widetilde{g}^{\mu\nu}}\approx d_\lambda\left(2\,\tau^\mu\widetilde{g}^{\lambda\alpha}\,\frac{\partial L_q}{\partial\widetilde{g}^{\alpha\mu}}+\tau^\lambda L_q-d_\mu U^{\mu\lambda}\right), \tag{56}$$

where

$$U=2\,\frac{\partial L_{AM}}{\partial R_{\mu\lambda}{}^\alpha{}_\nu}\,(\partial_\nu\tau^\alpha-k_\sigma{}^\alpha{}_\nu\tau^\sigma),$$

is the generalized Komar superpotential.[3,23] If $L_q=0$, we obtain exactly the energy-momentum conservation law of the metric-affine gravitation theory, but with an effective metric. In the case of the Logunov's Lagrangian

$$L_q=\lambda_3 g_{\mu\nu}\widetilde{\sigma}^{\mu\nu}\sqrt{|\widetilde{\sigma}|},$$

the equality (56) comes to the well-known condition

$$\nabla_\alpha(\widetilde{g}^{\alpha\mu}\sqrt{|\widetilde{g}|})\approx 0,$$

where ∇_α are covariant derivatives relative to the Levi-Civita connection of the background metric g.

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