GAUGE GRAVITATION THEORY

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PREFACE

Gauge theory is generally recognized to provide us with the adequate picture of the fundamental interactions. The gauge approach to the gravitation interaction establishes two main features of gravity as a physical field.

Gravitation phenomena are described by two geometric fields. These are an Einstein (tetrad or metric) gravitational field and a Lorentz connection. A Lorentz connection plays the role of a gauge gravitational potential induced by a gauge potential of fermion fields.

An Einstein gravitational field is a Higgs field which accompanies the spontaneous breaking of world symmetries. This spontaneous breaking takes place because of the coexistence of the Dirac fermion matter with exact Lorentz symmetries and the world geometric arena. Moreover, in contrast with Higgs fields of the Grand Unification models, Einstein gravitational field is a macroscopic Higgs field due to the peculiarity of gauge world transformations.

This book concerns only the gauge theory of the classical gravity. In the algebraic quantum theory, Higgs fields characterize nonequivalent Gaussian states on the algebras of quantum fields. They are "fictitious" fields describing collective phenomena. These fields fail to be quantized in the framework of the conventional quantum field theory. The Higgs nature of gravity therefore may open the door to many unexpected quantum effects.

Our formulation of gauge theory uses the machinery of modern differential geometry. Preliminary and Chapter 1 of this book are intended as an introduction to the jet bundle formalism and to the geometric theory of classical fields. In this book, we consider those aspects of gauge theory which explain local phenomena, although the theory itself is formulated in global terms.

INTRODUCTION

The geometric nature of classical gravity as a metric field has been established by Einstein's General Relativity. Its physical feature as a Higgs-Goldstone field corresponding to spontaneous breakdown of world symmetries is clarified owing to the gauge reformulation of gravitation theory in fibre bundle terms. Thus, gravity joins the unified gauge picture of the fundamental interactions.

The main problem of the gauge gravitation theory consists in that an Einstein gravitational field is a metric (or tetrad) field, whereas gauge potentials are connections. To settle this dilemma, many authors attempted to use the seeming identity of the tensor ranks of tetrad functions h^a_μ and gauge potentials σ^ν_μ of the translation subgroup of the Poincaré group (Section 4.1). They lost sight of Higgs-Goldstone fields appearing in gauge models due to spontaneous symmetry breaking. Moreover, the standard Yang-Mills scheme of gauge theory based on replacing global symmetries by the local ones appeared to be unsatisfactory for gauge theory of world symmetries. For instance, the holonomic transformations fail to be reproduced in this way. Besides, there are different types of gauge transformations (atlas transformations, principal morphisms, gauge freedom transformations etc.) which the conventional gauge principle fails to discern.

We therefore are based directly on the fibre bundle reformulation of classical field theory (Section 1.1). The necessary mathematical machinery can be exhausted by references [KOB, SUL, SAU, MAN 1991].

In bundle terms, classical fields are described by sections ϕ of some differentiable bundle E over a world manifold X. To construct differential operators and field Lagrangians one may use the jet bundle formalism. In its framework, a Lagrangian density $\mathcal{L}$ *of fields* ϕ *is defined on the 1-jet manifold* J^1E *of the bundle* E. Elements of J^1E are equivalence classes $j^1_x\phi$ of sections ϕ possessing the same values $\phi(x)$ and the same values of their first derivatives $\partial_\mu\phi(x)$ at $x \in X$. Given the world coordinates (x^μ) on X and the bundle coordinates (x^μ, y^i) on E, the jet manifold J^1E of the bundle E is endowed with the so-called adapted coordinates

$$(x^\lambda, y^i, y^i_\lambda) \circ j^1\phi = (x^\lambda, \phi^i(x), \partial_\lambda\phi^i(x)).$$

The jet manifold plays the role of a finite-dimensional configuration space of classical fields ϕ.

The corresponding finite-dimensional momentum space of fields ϕ is represented by the Legendre manifold Π provided with the so-called standard coordinates $(x^\lambda, y^i, p^\lambda_i)$ (Section 1.3). Given a Lagrangian density $\mathcal{L}$, we have the Legendre morphism

$$(x^\lambda, y^i, y^i_\lambda) \to (x^\lambda, y^i, p^\lambda_i = \partial^\lambda_i \mathcal{L}).$$

and the multimomentum Hamiltonian

$$\mathcal{H}(x^\lambda, y^i, p^\lambda_i) = p^\lambda_i y^i_\lambda - \mathcal{L}.$$

Let us remark that, in the multimomentum Hamiltonian formalism, time and spatial coordinates are considered on the same footing and so, this machinery does not require the preliminary (3+1) decomposition of a world manifold.

The jet bundle formalism enables us to manipulate general connections defined as sections Γ of the bundle $J^1E \to E$. Principal connections keep their physical importance because, to construct a gauge invariant Lagrangian, one must reduce the bundle E to the one associated with some principal bundle. We use general connections in the models of spontaneous symmetry breaking and in the multimomentum Hamiltonian formalism. For instance, there is the canonical splitting of a multimomentum Hamiltonian

$$\mathcal{H} = p^\lambda_i \Gamma^i_\lambda(y) + \widetilde{\mathcal{H}}$$

where Γ is some general connection on the bundle E.

In fibre bundle form, the gauge principle is reduced to the natural requirement of Lagrangians (or multimomentum Hamiltonians) be invariant under transformations of the adapted coordinates on the configuration space J^1E (or the standard coordinates on the Legendre bundle Π). Such coordinates are induced by atlases of the bundle E and the tangent bundle TX over a world manifold X. In field theory, these atlases define internal and world reference frames. The gauge principle thus makes the sense of a relativity principle.

To construct a gauge invariant Lagrangian, one needs a metric a^E in fibres of the bundle E and a world metric g in fibres of the cotangent bundle T^*X. By gauge transformations, a fibre metric a^E can be always brought into a canonical form invariant under a structure group G of the bundle E. In contrast with a^E, a world metric g takes a canonical form η only with respect to nonholonomic atlases of T^*X in general and η is invariant only under some subgroup of world symmetries. It means that, in gauge theory of world symmetries there is a dynamic metric field besides a gauge potential Γ.

The relativity principle however does not require g be a pseudo-Riemannian metric and Γ be a gauge potential of the Lorentz group. One therefore needs a supplementary principle besides the gauge one in order to reduce gauge theory of world symmetries to the gauge gravitation theory. This is the equivalence principle.

In Einstein's General Relativity, the equivalence principle is called to guarantee the transition to the Special Relativity with respect to some reference frames. There exist various formulations of this principle. Most of them are corollaries of geometrization of a gravitational field by components of a pseudo-Riemannian metric. The equivalence principle that we need must result in the existence of a pseudo-Riemannian metric itself. In geometric terms, we have formulated this principle as follows [IVA 1983].

In the Minkowski space, a time coordinate parameterizes the set of events ordered by the genetic relations. Lorentz transformations describe the transformations of these relations under changing a reference frame. In the spirit of Klein's Erlanger program, the Minkowski space geometry can be characterized as the geometry of Lorentz invariants. The geometric equivalence principle then postulates that, with respect to some reference frames, Lorentz invariants can be defined everywhere on a world manifold X^4 and are preserved by parallel displacement. This principle has the adequate fibre bundle formulation. It requires that the principal linear frame bundle LX with the structure group

$$GL_4 = GL(4, \mathbb{R})$$

be reduced to some subbundle L^hX with the structure Lorentz group

$$L = SO(3,1)$$

and so, that a gravitational field h exist on a world manifold X^4 (Section 2.2). There is 1:1 correspondence between the reduced L-subbundles L^hX and the tetrad gravitational fields h represented by global sections of the LX-associated Higgs bundle Σ with the standard fibre GL_4/L. This bundle is isomorphic to the 2-fold covering of the bundle Λ of pseudo-Riemannian forms in cotangent spaces T^*_xX to X^4. A global section of Λ is a pseudo-Riemannian metric g on X^4. The geometric equivalence principle thereby provides X^4 with the so-called L-structure [SUL]. This means the following.

A principal connection Γ^h on the linear frame bundle LX is assumed to be an extension of some connection A on the reduced subbundle L^hX. A world manifold X^4 is a pseudo-Riemannian space with the metric g corresponding to the reduced subbundle L^hX. Atlases of L^hX are extended to the atlases Ψ^h of LX possessing Lorentz transition functions. With respect to Ψ^h, metric functions of g are reduced to the Minkowski metric η and the local connection form Γ^h_κ takes its values in the Lie algebra of the Lorentz group, that is, its coefficients represent components of a Lorentz gauge potential. We call Γ^h a Lorentz connection. It plays the role of a gauge gravitational potential. There is the canonical splitting of Γ^h in the sum

$$\Gamma^h = \{\,\} + S.$$

of the Christoffel symbols $\{\,\}$ of the metric g and the contortion form S. The gauge gravitation theory thereby is the theory of gravity with torsion in general [HEL, IVA 1983, OBU].

The geometric equivalence principle defines some space-time structure on a world manifold X^4 (Section 2.3). For every reduced subbundle L^hX, there exist reduced subbundles $L^{\mathbf{F}}X$ of LX with the structure group $SO(3) \subset L$. There is 1:1 correspondence between these subbundles and the smooth distributions $\mathbf{F}$ of

3-dimensional spatial subspaces of tangent spaces T_xX. Such a distribution yields the (3+1) decomposition of the tangent bundle TX over X^4 into the direct sum of the 3-dimensional spatial subbundle $\mathbf{F}$ and its time-like orthocomplement T^0X. This decomposition turns a world manifold into a space-time. In particular, some types of gravitation singularities can be described as singularities of space-time distributions.

The geometric equivalence principle singles out the Lorentz group as the exact symmetry subgroup of world symmetries broken spontaneously. The corresponding Higgs-Goldstone field is a classical metric (or tetrad) gravitational field.

Spontaneous symmetry breaking is the quantum phenomenon. It takes place if, given a symmetry group G and its subgroup H, a Gaussian state F on an algebra of matter fields is H-stable and nonequivalent to any G-stable state [SAR']. There are two types of spontaneous symmetry breaking:

(i) States Fg, $g \in G$, are equivalent to F.

(ii) States Fg, $g \in G$, are nonequivalent to F, e.g., if matter fields possess only the exact symmetry group H.

Spontaneous breaking of world symmetries belongs to the type (ii). The corresponding matter fields are Dirac fermion fields on which the Clifford algebra of Dirac's γ-matrices and the Dirac operator act. There are various spinor models of the fermion matter. For instance, infinite-dimensional representations of the group $SL(4,\mathbb{R})$ are examined [NEE 1985] and, in this case, the above-mentioned spontaneous breakdown of world symmetries takes no place. All observable fermion particles however are Dirac fermions.

Let E be a spinor bundle whose sections describe classical Dirac fermion fields ϕ on a world manifold X^4. There is an associated bundle E_M of Minkowski spaces with the structure Lorentz group so that the bundle morphism

$$\gamma_E\colon\ E_M \otimes E \to E$$

exists and defines representation of elements of E_M by Dirac's γ-matrices on elements of E. To define the Dirac operator on sections of E, one must require E_M be isomorphic to the cotangent bundle T^*X over a world manifold X^4. Since the structure group of T^*X is GL_4, it takes place only if there is some reduced L-subbundle L^hX of the linear frame bundle LX and E_M is associated with L^hX, that is, if the geometric equivalence principle holds. The cotangent bundle T^*X provided only with atlases Ψ^h possesses the structure of the Minkowski space bundle M^hX associated with the reduced subbundle L^hX. For different tetrad fields h and h', bundles M^hX and $M^{h'}X$ are not isomorphic to each other. Their fibres M_x and M'_x are cotangent spaces T^*_xX, but provided with different Minkowski space structures.

The peculiarity of gravitational field thus is clarified. In contrast to the other fields, a tetrad gravitational field itself defines reference frames and these reference frames corresponding to different gravitational field are nonequivalent in a sense.

Let the Minkowski space bundle E_M associated with a spinor bundle E be isomorphic to the bundle

$$T^*X \cong M^hX.$$

Then, one can define the representation

$$\gamma_h\colon\ M^hX \otimes E \to E,$$
$$\gamma_h(dx^\mu) = h^\mu_a(x)\gamma^a$$

of cotangent vectors to X^4 (that is, 1-forms) by Dirac's γ-matrices on sections of a spinor bundle E. We denote such a spinor bundle (endowed with the representation morphism γ_h) by E^h (Section 2.2). Sections of E^h describe Dirac fermion fields ϕ_h in the presence of the tetrad gravitational field h.

Moreover, each principal connection A_s on the spinor bundle E^h induces a certain principal connection A on the reduced subbundle L^hX of LX and A is uniquely extended to a Lorentz connection Γ^h on the linear frame bundle LX. In other words, gauge potentials A_s of fermion fields generate gauge gravitational potentials.

The Higgs character of gravity issues from the fact that different gravitational fields h and h' define the nonisomorphic representations γ_h and $\gamma_{h'}$. It follows that Dirac fermion fields must be considered only in a pair with a certain gravitational field. These pairs fail to be represented by sections of the bundle product $\Sigma \times E$ of the Higgs bundle Σ and some spinor bundle E, but form *sui generis* a fermion-gravitation complex (Section 3.1). To describe this complex, we use the fact that the total space P of the principal bundle LX represents the total space of the L-principal bundle P^L over the Higgs manifold

$$\Sigma = P/L.$$

The Higgs manifold Σ is parameterized both by coordinates x^μ of a world manifold X^4 and by values σ^μ_a which tetrad gravitational fields take in the quotient space GL_4/L. The manifold Σ is the finite-dimensional analogue of the Wheeler-DeWitt superspace in a sense.

Let $E^L \to \Sigma$ be a spinor bundle associated with P^L. Fermion-gravitation pairs can be represented by sections of the composite bundle

$$\widetilde{E} = E^L \to \Sigma \to X$$

over X. This bundle however is not associated with a principal bundle and so, does not admit a principal connection. To define a connection on $\widetilde{E}$, one uses principal connections on the bundles Σ and P^L and the canonical jet bundle morphism

$$J^1\Sigma \underset{\Sigma}{\times} J^1E^L \to J^1\widetilde{E}.$$

As a result, covariant derivatives of fermion fields include the additional term due to parallel displacement along the coordinates σ^μ_a of the Higgs bundle Σ.

Since, for different gravitational fields h and h', the representations γ_h and $\gamma_{h'}$ are not isomorphic, tetrad gravitational fields, unlike matter fields and gauge potentials, fail to form a linear space or an affine space modelled on a linear space of deviations from some background field. They thereby do not satisfy the superposition principle and can not be quantized by usual methods because, in accordance with the conventional quantum field theory, fields must form a linear space in order to be quantized.

This is the common feature of Higgs fields. In algebraic quantum field theory, different Higgs fields correspond to nonequivalent Gaussian states on an algebra of matter fields. Quantized deviations of a Higgs field can not change a Gaussian state of this algebra, and so fail to result in some new Higgs field. A Higgs field thereby is a classical field. If one considers its small classical deviations being superposable in some approximation, their quantums turn out to be quasi-particles, not true particles.

A classical tetrad gravitational field as a Higgs field also is "fictitious" in a sense. It describes a field of invariance relations which is preserved by parallel displacement. For instance, a momentum part of the multimomentum Hamiltonian form for the classical gravity is reduced only to the connection term. Thus, quantization of tetrad (metric) gravitational fields goes beyond the framework of the standard quantum field theory.

At the same time, one can examine superposable deviations σ of a tetrad gravitational field h such that

$$h+\sigma$$

is not a tetrad gravitational field (Section 3.2). For instance, they do not change atlases Ψ^h and the world metric g. These deviations are generated by non-Lorentz transformations of fibres of T^*X and thereby violate the isomorphism between E_M and T^*X. Such transformations look like deformations of a world manifold in the gauge theory of space-time translations (Section 4.2). A Lagrangian of superposable deviations σ differs from the familiar Lagrangians of gravitation theory. For instance, contains the mass-like terms.

In other words, the superposable deviations σ of a tetrad gravitational field can destroy the correlation between the Dirac fermion matter and the space-time geometric arena. At the same time, if world symmetries are not broken (e.g., there are no fermion fields), transmutations of $h+\sigma$ into a new gravitational field h' may take place and may be accompanied by violation of the usual energy conservation law.

In the Grand Unification models, appearance of a Higgs field is usually related to a phase transition. A gravitational field also might arise owing to some primary phase transition which had separated prematter and pregeometry.

PRELIMINARY

We assume that all morphisms are smooth (that is, of class C^∞) and manifolds are real, Hausdorff, finite-dimensional, and second-countable (as a consequence, paracompact). Unless otherwise stated, structure groups of bundles are real finite-dimensional Lie groups.

By $\wedge$, we denote exterior product (i.e., skew-symmetric tensor product) of cotangent vectors.

Interior product (pairing) of tangent vectors with cotangent vectors is denoted by $\rfloor$.

0.1 Bundles

By a bundle, we mean a locally trivial fibre bundle

$$\pi\colon E\to B$$

whose total space E and base B are manifolds. For the sake of simplicity, we denote a bundle by its total space symbol E.

We use y and x in order to denote points of E and B respectively.

Given a bundle E and another bundle

$$\pi'\colon E'\to B',$$

a bundle morphism of E to E' is defined to be a pair of manifold morphisms

$$\Phi\colon E\to E', \qquad \Phi_B\colon B\to B'$$

such that

$$\pi'\circ\Phi=\Phi_B\circ\pi.$$

One says that Φ is a bundle morphism over Φ_B. If

$$\Phi_B=\mathrm{id}\,B,$$

then Φ is called a bundle morphism over B.

Let E and E' be bundles over the same base B. We denote their bundle product over B by

$$E \underset{B}{\times} E'.$$

There are two bundle projections

$$\mathrm{pr}_1\colon E \underset{B}{\times} E' \to E, \qquad \mathrm{pr}_2\colon E \underset{B}{\times} E' \to E'.$$

Given a bundle E and a manifold morphism

$$\Phi_B\colon B' \to B,$$

the pull-back of E by Φ_B is defined to be the bundle

$$\Phi_B^* E = \{(y, x') \in E \times B';\ \pi(y) = \Phi_B(x')\}$$

with the base B' and projection

$$\Phi_B^* \pi\colon (y, x') \to x'.$$

In particular, each section e of E yields the pull-back section

$$\Phi_B^* e(x') = (e(\Phi_B(x')), x')$$

of $\Phi_B^* E$. There is the bundle morphism

$$\overset{\circ}{\Phi}_B\colon \Phi_B^* E \ni (y, x') \to y \in E. \tag{0.1}$$

We provide a bundle E with local bundle coordinates

$$(x^\lambda, y^i), \quad 1 \leq \lambda \leq n = \dim B,$$
$$1 \leq i \leq l = \dim E - \dim B,$$

which are compatible with the bundle fibration of E, that is,

$$\mathrm{pr}_1 \circ (x^\lambda, y^i) = x^\lambda \circ \pi.$$

In particular, if

$$\Psi = \left\{U_\kappa, \psi_\kappa\colon \pi^{-1}(U_\kappa) \to U_\kappa \times F\right\}$$

is a bundle atlas of E, coordinates y^i on E can be induced by coordinates v^i on a standard fibre F of the bundle E:

$$y^i = v^i \circ \psi_\kappa. \tag{0.2}$$

We call coordinates (0.2) the bundle coordinates associated with an atlas Ψ.

In field theory, one is usually concerned with bundles possessing additional algebraic structure.

A group bundle is defined to be a bundle E together with canonical bundle morphisms m and k over B and a global section e_E of E:

$$\begin{aligned} &m\colon E \underset{B}{\times} E \to E,\\ &k\colon E \to E,\\ &e_E\colon B \to E. \end{aligned} \tag{0.3}$$

They make each fibre

$$E_x = \pi^{-1}(x)$$

of the bundle E into a Lie group:

$$m(e_E(x), y) = m(y, e_E(x)) = y,$$
$$m(k(y), y) = m(y, k(y)) = e_E(x), \quad y \in E_x.$$

For instance, a vector bundle E possesses the structure of an additive group bundle. In this case, e_E is the canonical zero section of E.

A general affine bundle is defined to be a triple (E, E', r) of a bundle E, a group bundle E' over B, and a bundle morphism

$$r\colon E \underset{B}{\times} E' \to E$$

which makes each fibre E_x of E into a general affine space with the associated group E'_x acting freely and transitively on E_x.

In particular, if a group bundle is a vector bundle $\overline{E}$, a general affine bundle is called an affine bundle modelled on a vector bundle $\overline{E}$:

$$r_E\colon E \underset{B}{\times} \overline{E} \to E,$$
$$r_E\colon (y, \overline{y}) \to y + \overline{y}.$$

For instance, every vector bundle E can be provided with the canonical structure of an affine bundle (of translations in E) modelled on E by means of the morphism

$$r_E\colon (y, y') \to y + y'.$$

A principal bundle $P \to B$ with a structure group G is defined to be a general affine bundle with respect to the trivial group bundle $B \times G$ where the group G acts on P on the right:

$$r_g\colon P \to Pg, \qquad g \in G. \tag{0.4}$$

A standard fibre of a principal bundle P is its structure group G which acts on itself on the left. Fibres of P are diffeomorphic to the group space of G, but fail to be groups.

A principal bundle P is a general affine bundle also with respect to the principal group bundle $\widetilde{P}$. This is the P-associated bundle with the standard fibre G on which the structure group acts by the adjoint representation

$$\mathrm{ad}\, g\colon G \to gGg^{-1}, \qquad g \in G.$$

Fibres of $\widetilde{P}$ are groups isomorphic (but not canonically isomorphic) to the structure group G. Moreover, for any P-associated bundle E, the canonical bundle morphism

$$\widehat{P}_E\colon E \underset{B}{\times} \widetilde{P} \to E \tag{0.5}$$

is defined.

Remark. Given a principal bundle

$$\pi_P\colon P \to B$$

with a structure group G, a total space of a P-associated bundle E with a standard fibre F is defined to be the quotient $(P \times F)/G$ of the product $P \times F$ by identification of elements (p, v) and $(pg, g^{-1}v)$ for all $g \in G$. A global section e of E then is determined by a F-valued equivariant function f_e on P such that

$$e(\pi_P(p)) = [p]_F f_e(p), \qquad p \in P,$$
$$f_e(pg) = g^{-1} f_e(p), \qquad g \in G,$$

where $[p]_F$ denotes the restriction of the canonical map

$$P \times F \to E$$

to the subset $p \times F$.

Let (E_1, E_1', r_1) and (E_2, E_2', r_2) be general affine bundles. An affine bundle morphism $E_1 \to E_2$ is a pair of bundle morphisms

$$\Phi\colon E_1 \to E_2, \qquad \Phi'\colon E_1' \to E_2'$$

such that

$$r_2 \circ (\Phi, \Phi') = \Phi \circ r_1.$$

For instance, let $P \to B$ and $P' \to B'$ be principal bundles with a structure group G. Then, an affine bundle morphism of P to P' is defined to be a G-equivariant bundle morphism

$$(\Phi, \Phi' = (\Phi_B, \mathrm{id}\, G))$$

over a manifold morphism

$$\Phi_B\colon B \to B'$$

such that, whenever $g \in G$,

$$r_g' \circ \Phi = \Phi \circ r_g.$$

Every principal isomorphism of a principal bundle P (over the identity morphism of its base B) is expressed as

$$\Phi_P(p) = p f_s(p), \qquad p \in P, \tag{0.6}$$
$$f_s(pg) = g^{-1} f_s(p) g, \quad g \in G,$$

where f_s is a G-valued equivariant function on P corresponding to some global section s of the principal group bundle $\widetilde{P}$. Principal isomorphisms thus form the gauge group $G(B)$ which is canonically isomorphic to the group of global sections of the bundle $\widetilde{P}$.

Remark. There is no canonical embedding of G into $G(B)$ even if P is a trivial bundle. Elements of $G(B)$ take their values in fibres of the group bundle $\widetilde{P}$, but not in its standard fibre G.

Given a P-associated bundle E with a standard fibre F, every principal isomorphism (0.6) yields the associated principal morphism

$$\Phi_E\colon (P \times F)/G \to (\Phi_P(P) \times F)/G \tag{0.7}$$

of the bundle E so that

$$\Phi_E = \widehat{P}_E|_{E \times_B s(B)}.$$

Given affine bundles E and E' modelled on vector bundles $\overline{E}$ and $\overline{E}'$ respectively, a bundle morphism

$$\Phi\colon E \to E'$$

is affine if there exists a linear bundle morphism

$$\overline{\Phi}\colon \overline{E} \to \overline{E}'$$

satisfying the following condition

$$r_{E'} \circ (\Phi, \overline{\Phi}) = \Phi \circ r_E.$$

This linear bundle morphism $\overline{\Phi}$ is called the fibred derivative of Φ:

$$\Phi(y^i) + \overline{\Phi}(\overline{y}^i) = \Phi(y^i + \overline{y}^i). \tag{0.8}$$

Let E be a vector bundle. Bundle coordinates (x^λ, y^i) on E are called linear if functions y^i are linear on each fibre.

Let E be an affine bundle modelled on a vector bundle $\overline{E}$. Bundle coordinates (x^λ, y^i) on E are called affine if functions y^i are affine on each fibre. By taking their fibred derivatives, one obtains the linear bundle coordinates $(x^\lambda, \overline{y}^i)$ on $\overline{E}$:

$$\overline{y}^i(\overline{y}) = y^i(y + \overline{y}) - y^i(y).$$

If E is a vector bundle provided with the canonical structure of an affine bundle, we have

$$y^i(y) = \overline{y}^i(y).$$

Henceforth, when we deal with a vector bundle and an affine bundle modelled on a vector bundle, we shall refer to the linear bundle coordinates and to the affine bundle coordinates respectively.

Let us note that additional algebraic structure puts constraints on a bundle E. For instance, a bundle E with a standard fibre F can be regarded as a bundle with the structure group Diff F of all diffeomorphisms of F. If E is assumed to be associated with a G-principal bundle P, it means that the structure group Diff F of E is reducible to G and that only atlases of E associated with atlases of the principal bundle P are allowed. One must discern affine bundles and affine bundles with an affine structure group. Jet bundles described in Section 0.2 exemplify affine bundles which are not associated with an affine principal bundle.

Remark. Given a principal bundle P and a P-associated bundle E, we say that a bundle atlas

$$\Psi^P = \left\{U_\kappa, \psi^P_\kappa\right\}$$

of P and a bundle atlas

$$\Psi = \{U_\kappa, \psi_\kappa\}$$

of E are associated atlases if they are determined by the same family $\{z_\kappa(x), x \in U_\kappa\}$ of local sections of P, that is,

$$\psi^P_\kappa(z_\kappa(x)) = 1_G,$$
$$\psi_\kappa(x) = [z_\kappa(x)]^{-1}_F, \quad \pi_P(p) = x \in U_\kappa,$$
$$z_\kappa(p) = z_\nu(p)\rho_{\nu\kappa}(\pi_P(p)), \quad \pi_P(p) = x \in U_\kappa \cup U_\nu,$$
$$\rho_{\kappa\nu}(x) = \psi_\kappa(x)\psi_\nu{}^{-1}(x).$$

Here, $\rho_{\kappa\nu}$ are G-valued transition functions of atlases Ψ^P and Ψ and 1_G is the unit element of the group G. By $\psi_\kappa(x)$, we denote the morphism $\mathrm{pr}_2 \circ \psi_\kappa$ restricted to a fibre $\pi^{-1}(x)$:

$$\psi_\kappa(x)\colon \pi^{-1}(x) \to F.$$

The tangent bundle over a bundle E possesses additional structure which is the vertical subbundle.

Remark. Given the tangent bundle

$$\pi_M\colon TM \to M$$

and the cotangent bundle T^*M over a manifold M, we denote the familiar induced coordinates on TM and T^*M by $(x^\lambda, \dot{x}^\lambda)$ and $(x^\lambda, \dot{x}_\lambda)$ respectively. Here, $\dot{x}^\lambda$ and $\dot{x}_\lambda$ are the coordinates on fibres T_xM and T^*_xM with respect to their holonomic bases $\{\partial_\lambda\}$ and $\{dx^\lambda\}$. Let

$$\Phi\colon M \to N$$

be a manifold morphism. It gives rise to the following linear bundle morphism over Φ:

$$\Phi_*\colon TM \to TN,$$
$$\Phi_*\colon \tau^\mu\partial_\mu \to \tau^\mu\frac{\partial\Phi^\nu}{\partial x^\mu}\partial'_\nu,$$

which is called the tangent morphism to Φ.

Given a bundle E, we have the tangent bundle

$$\pi_E\colon TE \to E$$

and the bundle

$$\pi_*\colon TE \to TB.$$

Given the bundle coordinates (x^λ, y^i) on E, the induced coordinates on TE are

$$(x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i).$$

The vertical bundle over a bundle E is defined to be the subbundle

$$VE = \ker \pi_* \subset TE.$$

The induced bundle coordinates on VE are

$$(x^\lambda, y^i, \dot{y}^i).$$

We have the following exact sequence of tangent bundles over E:

$$0 \to VE \to TE \to E \underset{B}{\times} TB \to 0 \tag{0.9}$$

where

$$E \underset{B}{\times} TB = \pi^*(TB)$$

is the pull-back of the tangent bundle TB by π. For instance, a bundle morphism Φ of a bundle E to E' gives rise to the vertical tangent morphism

$$V\Phi = \Phi_*|_{VE}\colon VE \to VE'$$

of the vertical bundle VE to VE'.

The dual exact sequence of cotangent bundles is

$$0 \to \pi^*(T^*B) \to T^*E \to V^*E \to 0. \tag{0.10}$$

Here, V^*E is the vertical cotangent bundle dual to VE and

$$H^*E = \pi^*(T^*B) = E \underset{B}{\times} T^*B$$

is the horizontal cotangent subbundle of T^*E which consists of covectors whose interior product with vertical tangent vectors is equal to zero. For the sake of simplicity, we usually denote the horizontal cotangent subbundle H^*E by T^*B.

By $\mathcal{T}(M)$, we denote the sheaf of vector fields

$$u\colon M \to TM$$

on a manifold M.

A vector field $u \in \mathcal{T}(E)$ on a bundle E is called a projectable vector field if it is projected to a vector field $u_B \in \mathcal{T}(B)$ on the base B. The coordinate expression of a projectable vector field is given by

$$u = u^\mu(x)\partial_\mu + u^i(y)\partial_i. \tag{0.11}$$

We denote the subsheaf of projectable vector fields by

$$\mathcal{P}(E) \subset \mathcal{T}(E).$$

A projectable vector field on E taking its values in the vertical bundle VE is called a vertical vector field. Its coordinate expression reads

$$u = u^i(y)\partial_i.$$

We denote the subsheaf of vertical vector fields by

$$\mathcal{V}(E) \subset \mathcal{P}(E) \subset \mathcal{T}(E).$$

Vertical bundles for the most of bundles relevant for physics possess simple structure called the vertical splitting.

Vertical splitting of a bundle E is constituted by some vector bundle $\overline{E}$ and a linear bundle isomorphism over E:

$$\alpha\colon VE \to E \underset{B}{\times} \overline{E}. \tag{0.12}$$

In particular, trivial vertical splitting of a bundle E is the vertical splitting with a trivial vector bundle $\overline{E} = B \times \overline{F}$:

$$\alpha\colon VE \to E \times \overline{F}. \tag{0.13}$$

Given the vertical splitting (0.12), bundle coordinates (x^λ, y^i) on E are called the coordinates adapted to vertical splitting if the vector fields

$$\mathrm{pr}_2 \circ \alpha \circ \partial_i\colon E \to VE \to E \underset{B}{\times} \overline{E} \to \overline{E}$$

are constant along fibres of E. In this case, we can write

$$\mathrm{pr}_2 \circ \alpha \circ \partial_i = t_i(x).$$

Here, $t_i(x)$ are bases for fibres of the vector bundle $\overline{E}$ which are associated with some local trivialization ψ of $\overline{E}$, that is,

$$\{t_i(x)\} = \left\{\psi^{-1}(t_i)\right\} \tag{0.14}$$

where $\{t_i\}$ is a fixed basis for a standard fibre $\overline{F}$ of the vector bundle $\overline{E}$. The induced coordinates on VE then read

$$\left(x^\mu, y^i, \dot{y}^i = \overline{y}^i \circ \alpha\right)$$

where $(x^\mu, \overline{y}^i)$ are bundle coordinates on $\overline{E}$. The vertical splitting (0.12) is called the integrable vertical splitting if there exists a bundle coordinate atlas of E constituted by coordinate charts adapted to the vertical splitting.

For instance, a vector bundle E has the canonical integrable vertical splitting

$$VE = E \times E. \tag{0.15}$$

An affine bundle E modelled on a vector bundle $\overline{E}$ has the canonical integrable vertical splitting (0.12). Linear bundle coordinates on a vector bundle and affine bundle coordinates on an affine bundle are adapted to these vertical splittings.

A principal bundle P with a structure group G has the canonical trivial vertical splitting (0.13):

$$\begin{aligned} &\alpha\colon VP \to P \times \mathfrak{g}, \\ &\mathrm{pr}_2 \circ \alpha \circ \partial_m = J_m, \end{aligned} \tag{0.16}$$

where $\mathfrak{g}$ is the left Lie algebra of the group G and $\{J_m\}$ is a basis for $\mathfrak{g}$. This splitting takes place because, by definition, elements of the left Lie algebra $\mathfrak{g}$ are left-invariant vector fields on G. Given an atlas $\{z_\kappa\}$ of the bundle P, the canonical bundle coordinates on P adapted to the canonical vertical splitting (0.16) are (x^λ, p^m):

$$p^m(p) = (a^m \circ \psi^P_\kappa)(p) = a^m(g_p), \qquad p \in \pi_P^{-1}(U_\kappa), \tag{0.17}$$

where $a^m(g)$ are the group parameters of an element g and the element $g_p \in G$ is determined by the relation

$$p = z_\kappa(\pi_P(p))g_p.$$

Note that tangent bundles over structured bundles are also structured bundles owing to tangent prolongation of corresponding morphisms.

Let E be a group bundle and the bundle

$$\pi_*: TE \to TB \tag{0.18}$$

be provided with morphisms m_*, k_*, and $(e_E)_*$ tangent to morphisms (0.3). The bundle TE then is a group bundle. For instance, if E is a vector bundle, the bundle (0.18) also is a vector bundle.

For bundles, the familiar machinery of $\mathbb{R}$-valued exterior forms is generalized to tangent-valued forms.

An exterior r-form ω on a manifold M is defined to be a section of the skew-symmetric tensor bundle $\overset{r}{\wedge} T^*M$. It has the coordinate expression

$$\omega = \omega_{\lambda_1\dots\lambda_r}(x)dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_r}.$$

The exterior differential d and exterior product $\wedge$ of exterior forms are defined in a usual way.

Interior product of a vector field $\tau = \tau^\mu\partial_\mu$ and a r-form ω reads

$$\tau\rfloor\omega = \sum_{p=1}^{r}(-1)^{p-1}\tau^\mu\omega_{\lambda_1\dots\lambda_{p-1}\mu\lambda_{p+1}\dots\lambda_r}(x)$$
$$\times dx^{\lambda_1}\wedge\cdots\wedge x^{\lambda_{p-1}}\wedge dx^{\lambda_{p+1}}\wedge\cdots\wedge dx^{\lambda_r}.$$

In particular, given a r-form ω, we introduce the (r–s)-form

$$\omega_{\lambda_1\dots\lambda_s} = \partial_{\lambda_1}\rfloor\cdots\rfloor(\partial_{\lambda_s}\rfloor\omega), \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{0.19}$$

Remark. A fibre metric g in the tangent bundle TM is defined to be a nondegenerate global section of the symmetric tensor bundle $\overset{2}{\vee} T^*M$, that is,

$$\det|g_{\mu\nu}(x)| \neq 0,$$
$$g(x) = g_{\mu\nu}(x)dx^\mu\otimes dx^\nu, \qquad x\in M.$$

For the sake of simplicity, we denote the dual fibre metric in the cotangent bundle T^*M by the same symbol g and simply call g a fibre metric on a manifold M.

Let a manifold M be oriented and provided with a fibre metric g. Let $\{dx^\lambda\}$ be the bases for T^*M which are compatible with a given orientation. One defines the volume form

$$\mathcal{V} = \sqrt{|g|}\,dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_m}, \qquad g = \det|g_{\mu\nu}|, \tag{0.20}$$

and the Hodge operator

$$*:\ \omega\to *\omega = g^{\lambda_1\mu_1}\dots g^{\lambda_r\mu_r}\omega_{\lambda_1\dots\lambda_r}(x)\mathcal{V}_{\mu_1\dots\mu_r}.$$

We denote the sheaf of exterior r-forms on a manifold M by $\overset{r}{\wedge} \mathcal{T}^*(M)$.

A tangent-valued r-form ϕ on a manifold M is defined to be a section of the bundle

$$\overset{r}{\wedge} T^*M \underset{M}{\otimes} TM.$$

It has the coordinate expression

$$\phi = \phi^\mu_{\lambda_1\dots\lambda_r}(x)\partial_\mu\otimes dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_r}.$$

In particular, tangent-valued 0-forms are vector fields on M. One introduces the canonical tangent-valued 1-form θ_M on a manifold M whose coordinate expression is given by

$$\theta_M = \partial_\lambda\otimes dx^\lambda. \tag{0.21}$$

Tangent-valued forms on a manifold M constitute the sheaf

$$\wedge\mathcal{T}^*(M)\otimes\mathcal{T}(M).$$

Given a bundle E, one can consider the following classes of tangent-valued forms on E:

(i) tangent-valued horizontal forms

$$\phi:\ E\to\wedge T^*B\underset{E}{\otimes}TE,$$
$$\phi = (\phi^\mu_{\lambda_1\dots\lambda_r}(y)\partial_\mu + \phi^i_{\lambda_1\dots\lambda_r}(y)\partial_i)\otimes dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_r},$$

(ii) projectable horizontal forms projected to tangent-valued forms on the base B:

$$\phi = (\phi^\mu_{\lambda_1\dots\lambda_r}(x)\partial_\mu + \phi^i_{\lambda_1\dots\lambda_r}(y)\partial_i)\otimes dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_r},$$

(iii) vertical-valued horizontal forms

$$\phi:\ E\to\wedge T^*B\underset{E}{\otimes}VE,$$
$$\phi = \phi^i_{\lambda_1\dots\lambda_r}(y)\partial_i\otimes dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_r}.$$

When a bundle E is endowed with the integrable vertical splitting (0.12), we can write

$$\wedge T^*B\underset{E}{\otimes}VE\cong E\underset{B}{\times}\left(\wedge T^*B\underset{B}{\otimes}\overline{E}\right).$$

Then, a vertical-valued horizontal r-form

$$\phi:\ E\to\overset{r}{\wedge} T^*B\underset{E}{\otimes}VE\cong E\underset{B}{\times}\left(\overset{r}{\wedge} T^*B\underset{B}{\otimes}\overline{E}\right)$$

satisfies the relation

$$\mathrm{pr}_2 \circ \alpha \circ \phi = \overline{\phi}$$

where $\overline{\phi}$ is some $\overline{E}$-valued horizontal form on E:

$$\overline{\phi}\colon E \to \overset{r}{\wedge} T^*B \underset{B}{\otimes} \overline{E}.$$

It has the coordinate expression

$$\begin{aligned}\phi &= \phi^i_{\lambda_1\ldots\lambda_r}(y)\partial_i \otimes dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r}\\ &= \overline{\phi}^i_{\lambda_1\ldots\lambda_r}(y)t_i(x) \otimes dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r}\end{aligned}$$

where $\{t_i(x)\}$ are bases (0.14) associated with some local trivialization ψ of the bundle $\overline{E}$.

Given integrable vertical splitting of a bundle E, a vertical-valued horizontal form on E is called a basic form if it is constant along fibres of E. The coordinate expression of a basic form reads

$$\phi = \overline{\phi}^i_{\lambda_1\ldots\lambda_r}(x)t_i(x) \otimes dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r}$$

where $\overline{\phi}$ is some $\overline{E}$-valued form on the base B.

A vertical-valued horizontal 1-form on a bundle E is called a soldering form

$$\sigma\colon E \to T^*B \underset{E}{\otimes} VE.$$

Its coordinate expression is given by

$$\sigma = \sigma^i_\lambda(y)\,\partial_i \otimes dx^\lambda. \tag{0.22}$$

In the particular case of a bundle E endowed with integrable vertical splitting, it makes sense to consider a basic soldering form

$$\sigma\colon B \to T^*B \underset{B}{\otimes} \overline{E}.$$

For example, let $E = TB$. Then $\overline{E} = TB$ and we have the canonical basic soldering form (0.21).

Tangent-valued 0-forms (that is, vector fields) are known to form the sheaf of Lie algebras with respect to the commutation brackets. This algebraic structure can be generalized to tangent-valued forms if we consider the Frölicher-Nijenhuis (F-N) brackets.

The F-N brackets are defined to be the sheaf morphism

$$\begin{aligned}[\,,\,]_{FN}\colon (\phi,\sigma) \to [\phi,\sigma]_{FN} &= [\alpha\otimes u, \beta\otimes v]_{FN}\\ &= \alpha\wedge\beta\otimes[u,v] + \alpha\wedge L_u\beta\otimes v - (-1)^{rs}\beta\wedge L_v\alpha\otimes u\\ &\quad + (-1)^r(v\rfloor\alpha)\wedge d\beta\otimes u - (-1)^{rs+s}(u\rfloor\beta)\wedge d\alpha\otimes v,\end{aligned}$$

$$\alpha \in \overset{r}{\wedge} T^*(M), \quad \beta \in \overset{s}{\wedge} T^*(M), \quad u,v \in \mathcal{T}(M),$$

where L_u and L_v are Lie derivatives. We have the coordinate expression

$$\begin{aligned}[\phi,\sigma]_{FN} = (&\phi^\nu_{\lambda_1\ldots\lambda_r}\partial_\nu\sigma^\mu_{\lambda_{r+1}\ldots\lambda_{r+s}} - (-1)^{rs}\sigma^\nu_{\lambda_1\ldots\lambda_s}\partial_\nu\phi^\mu_{\lambda_{s+1}\ldots\lambda_{r+s}}\\ &- r\phi^\nu_{\lambda_1\ldots\lambda_{r-1}\nu}\partial_{\lambda_r}\sigma^\nu_{\lambda_{r+1}\ldots\lambda_{r+s}} + (-1)^{rs}s\sigma^\mu_{\lambda_1\ldots\lambda_{s-1}\nu}\partial_{\lambda_s}\phi^\nu_{\lambda_{s+1}\ldots\lambda_{r+s}})\\ &dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_{r+s}}\otimes\partial_\mu.\end{aligned}$$

The sheaf $\wedge\mathcal{T}^*(M)\otimes\mathcal{T}(M)$ endowed with the F-N brackets thus is the sheaf of graded Lie algebras:

$$\begin{aligned}&[\phi,\sigma]_{FN} = -(-1)^{|\phi||\sigma|}[\sigma,\phi],\\ &[\theta,[\phi,\sigma]_{FN}]_{FN} = [[\theta,\phi]_{FN},\sigma]_{FN} + (-1)^{|\theta||\phi|}[\phi,[\theta,\sigma]_{FN}]_{FN},\\ &\phi,\sigma,\theta \in \wedge\mathcal{T}^*(M)\otimes\mathcal{T}(M),\end{aligned}$$

where $|\phi|$ denotes the degree of a form ϕ.

Given a tangent-valued form

$$\theta \in \overset{r}{\wedge} \mathcal{T}^*(M)\otimes\mathcal{T}(M),$$

we can introduce the Nijenhuis differential

$$d_\theta\colon \sigma \mapsto d_\theta\sigma = [\theta,\sigma]_{FN}. \tag{0.23}$$

For example, if $\theta = u$ is a vector field, we obtain the Lie derivative

$$\begin{aligned}d_u\sigma = L_u\sigma = \big(&u^\nu\partial_\nu\sigma^\mu_{\lambda_1\ldots\lambda_s} - \sigma^\nu_{\lambda_1\ldots\lambda_s}\partial_\nu u^\mu\\ &+ s\sigma^\mu_{\lambda_1\ldots\lambda_{s-1}\nu}\partial_{\lambda_s}u^\nu)dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_s}\otimes\partial_\mu.\end{aligned} \tag{0.24}$$

Note that the differential (0.23) can be applied to $\mathbb{R}$-valued forms σ.

Because of the Jacoby identity, d_θ turns out to be the r-degree derivation of the F-N algebra $\wedge\mathcal{T}^*(M)\otimes\mathcal{T}(M)$. Namely, we have

$$\begin{aligned}&d_\theta[\phi,\sigma]_{FN} = [d_\theta\phi,\sigma]_{FN} + (-1)^{|\theta||\phi|}[\phi,d_\theta\sigma]_{FN},\\ &d_\theta d_\eta - (-1)^{|\eta||\theta|}d_\eta d_\theta = d_{[\theta,\eta]_{FN}}.\end{aligned}$$

In particular, for $\theta = \eta$, we obtain

$$\frac{1}{2}(1-(-1)^r)d^2_\theta\phi = d_{K_\theta}\phi = [K_\theta,\phi]_{FN}$$

where

$$K_\theta = \frac{1}{2}d_\theta\theta \in \overset{2r}{\wedge} \mathcal{T}^*(M)\otimes\mathcal{T}(M) \tag{0.25}$$

is called the curvature of a tangent-valued form θ. Note that

$$K_\theta = 0$$

for even r. For odd r, we have the generalized Bianchi identity

$$d_\theta K_\theta = 0. \tag{0.26}$$

Given a fibre metric g, one can introduce the covariant codifferential defined to be the sheaf morphism

$$\delta_\theta = -(-1)^{m(r-i)+i} * d_\theta *\colon \overset{s}{\wedge} T^*(M) \otimes T(M) \to \overset{s-r}{\wedge} T^*(M) \otimes T(M) \tag{0.27}$$

where $m = \dim M$ and i is the signature of the metric g.

0.2 Jet Manifolds

We here restrict ourselves to the first and second order jet manifolds.

Given a bundle E, the first order jet manifold J^1E of E is defined to comprise classes

$$w = j^1_x e, \qquad x \in B,$$

of sections e of E so that sections e and e' belong to the same class $j^1_x e$ if and only if

$$e(x) = e'(x), \qquad e_*|_{T_xB} = e'_*|_{T_xB}.$$

The jet manifold J^1E is the total space both of the bundle

$$E^1 = (J^1E, \pi_1, E), \qquad \pi_1\colon J^1E \ni j^1_x e \to x \in B, \tag{0.28}$$

and of the bundle

$$E^{01} = (J^1E, \pi_{01}, E), \qquad \pi_{01}\colon J^1E \ni j^1_x e \to e(x) \in E. \tag{0.29}$$

Note that the structure of a smooth finite-dimensional manifold is induced on J^1E as on the bundle E^{01}.

Given the bundle coordinates (x^λ, y^i) on E, the jet manifold J^1E is provided with the following local coordinates $(x^\lambda, y^i, y^i_\lambda)$ called the adapted coordinates:

$$\begin{aligned}
x^\lambda(w) &= x^\lambda(j^1_x e) = x^\lambda(x),\\
y^i(w) &= y^i(j^1_x e) = y^i(e(x)) = e^i(x),\\
y^i_\lambda(w) &= y^i_\lambda(j^1_x e) = \partial_\lambda y^i(e(x)) = e_*{}^i_\lambda(x).
\end{aligned}$$

Adapted coordinate transformations read

$$x'^\lambda = \Phi^\lambda_B(x^\mu), \tag{0.30a}$$
$$y'^i = \Phi^i(x^\mu, y^j), \tag{0.30b}$$
$$y'^i_\lambda = \left(\frac{\partial \Phi^i}{\partial y^j} y^j_\mu + \frac{\partial \Phi^i}{\partial x^\mu}\right)\frac{\partial x^\mu}{\partial x'^\lambda}. \tag{0.30c}$$

Let us point out the fact that transformation law (0.30a) is independent of y^i and y^i_λ and that transformation law (0.30b) does not involve y^i_λ. It follows that the adapted coordinates on J^1E are the bundle coordinates both of the bundle E^1 and of the bundle E^{01}.

Moreover, the second term in transformation law (0.30c) indicates the fact that E^{01} is an affine bundle. Namely, there is the canonical bundle monomorphism θ_1 of J^1E onto an affine subbundle of the bundle

$$T^*B \underset{E}{\otimes} TE \to E.$$

This monomorphism is called the contact map and is given by the coordinate expression

$$\theta_1 = dx^\lambda \otimes d_\lambda = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i). \tag{0.31}$$

The bundle E^{01} hence is the affine bundle modelled on the vector bundle

$$T^*B \underset{E}{\otimes} VE \to E.$$

There is another canonical bundle monomorphism θ_2 of J^1E onto an affine subbundle of the bundle

$$T^*E \underset{E}{\otimes} VE \to E.$$

This monomorphism is called the contact form and is given by the coordinate expression

$$\theta_2 = (dy^i - y^i_\lambda dx^\lambda) \otimes \partial_i. \tag{0.32}$$

The contact map θ_1 and the contact form θ_2 yield canonical morphisms

$$\begin{aligned}
&\widehat{\theta}_1\colon\ \pi^*_{01}(TE) = TE \underset{E}{\times} J^1E \to TE\rfloor(\theta_1 J^1E) = HE \subset TE,\\
&\widehat{\theta}_2\colon\ TE \underset{E}{\times} J^1E \to TE\rfloor(\theta_2 J^1E) = VE.
\end{aligned} \tag{0.33}$$

Their coordinate expressions read

$$\begin{aligned}
\widehat{\theta}_1 &= \dot{x}^\lambda(\partial_\lambda + y^i_\lambda \partial_i),\\
\widehat{\theta}_2 &= (\dot{y}^i - \dot{x}^\lambda y^i_\lambda)\partial_i.
\end{aligned}$$

Morphisms (0.33) define the canonical horizontal splitting of the pull-back $\pi_{01}^*(TE)$ of the tangent bundle TE over J^1E:

$$TE \underset{E}{\times} J^1E = HE \underset{J^1E}{\oplus} VE,$$
$$\dot{x}^\lambda\partial_\lambda + \dot{y}^i\partial_i = \dot{x}^\lambda(\partial_\lambda + y^i_\lambda\partial_i) \oplus (\dot{y}^i - \dot{x}^\lambda y^i_\lambda)\partial_i. \tag{0.34}$$

Let E and E' be bundles over B and

$$\Phi\colon E \to E'$$

be some bundle morphism over a diffeomorphism Φ_B of B. Then, there exists the jet prolongation of the morphism Φ to the jet manifold morphism

$$j^1\Phi\colon J^1E \ni j^1_x e \to j^1_{\Phi_B(x)}(\Phi \circ e \circ \Phi_B^{-1}) \in J^1E'.$$

For instance, each section e of a bundle E can be regarded as a bundle morphism of the bundle $B \to B$ into the bundle E over B. Hence, we have the jet prolongation of a section e to the section

$$(j^1e)(x) = j^1_x e$$

of the bundle E^1. In the adapted coordinates, this prolongation reads

$$(x^\lambda, y^i, y^i_\lambda) \circ j^1e = (x^\lambda, e^i(x), \partial_\lambda e^i(x)).$$

Algebraic structure of a bundle E also has jet prolongations to the bundle E^1 owing to jet prolongation of corresponding morphisms.

Let E be a group bundle. The bundle E^1 provided with the jet prolongations j^1m, j^1k and j^1e_E of morphisms (0.3) becomes a group bundle. For instance, if E is a vector bundle, E^1 also is a vector bundle.

Let (E, E', r) be a general affine bundle. Being provided with the jet prolongation

$$j^1r\colon J^1E \underset{B}{\times} J^1E' \to J^1E,$$

the bundle E^1 is a general affine bundle with the group bundle E'^1. In particular, if E is an affine bundle modelled on a vector bundle $\overline{E}$, then E^1 is an affine bundle modelled on $\overline{E}^1$.

In Section 1.2, we shall need the lift of projectable vector field u (0.11) on a bundle E to a vector field

$$\overline{u}\colon J^1E \to TJ^1E$$

on the jet manifold J^1E. The coordinate expression of this lift is given by

$$\overline{u}(w) = u^\lambda\partial_\lambda + u^i\partial_i + (\partial_\lambda u^i + y^j_\lambda\partial_j u^i - y^i_\mu\partial_\lambda u^\mu)\partial_i^\lambda,$$
$$x = \pi(y) = \pi_1(w), \qquad y = \pi_{01}(w). \tag{0.35}$$

By analogy with the 1-jet manifold J^1E, the higher order jet manifolds J^kE can be introduced. We here concern only with the second order jet manifold J^2E

of a bundle E. This manifold comprises classes $j^2_x e$ of sections e of a bundle E so that sections e and e' belong to the same class $j^2_x e$ if and only if

$$j^1_x e = j^1_x e', \qquad e_{**}|_{TT_xB} = e'_{**}|_{TT_xB}.$$

By TTB, we here denote the tangent bundle over the tangent bundle TB.

Remark. There are two projections

$$\pi_T\colon TTB \to TB, \qquad \pi_{B*}\colon TTB \to TB.$$

In the induced coordinates $(x^\lambda, \dot{x}^\mu, \mathring{x}^\alpha, \ddot{x}^\beta)$ on TTB, projections π_T and π_{B*} read

$$(x^\lambda, \dot{x}^\mu) \circ \pi_T = (x^\lambda, \dot{x}^\mu), \qquad (x^\lambda, \dot{x}^\mu) \circ \pi_{B*} = (x^\lambda, \mathring{x}^\mu).$$

The 2-jet manifold J^2E is the total space of the following bundles:

$$E^2 = (\pi_2\colon J^2E \to B),$$
$$E^{12} = (\pi_{12}\colon J^2E \to J^1E),$$
$$E^{02} = (\pi_{02}\colon J^2E \to E).$$

This manifold is endowed with the adapted coordinates $(x^\lambda, y^i, y^i_\lambda, y^i_{\lambda\mu} = y^i_{\mu\lambda})$ where

$$y^i_{\lambda\mu}(j^2_x e) = \partial_\mu\partial_\lambda e^i(x).$$

One can generalize the canonical morphisms θ_1 (0.31) and θ_2 (0.32) to the 2-jet manifolds:

$$\theta_1\colon\ J^2E \to T^*B \underset{J^1E}{\otimes} TJ^1E,$$
$$\theta_2\colon\ J^2E \to VJ^1E \underset{J^1E}{\otimes} T^*J^1E$$

where T^*B denotes the pull-back $\pi_1^*(TB)$. Their coordinate expressions read

$$\theta_1 = dx^\lambda \otimes d_\lambda = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda\partial_i + y^i_{\lambda\mu}\partial_i^\mu),$$
$$\theta_2 = (dy^i - y^i_\mu dx^\mu) \otimes \partial_i + (dy^i_\lambda - y^i_{\lambda\mu}dx^\mu) \otimes \partial_i^\lambda. \tag{0.36}$$

Operators

$$d_\lambda = \partial_\lambda + y^i_\lambda\partial_i + y^i_{\lambda\mu}\partial_i^\mu \tag{0.37}$$

are called the total derivatives.

For instance, using expressions (0.36), we obtain the canonical decomposition of the exterior differential d into the vertical and horizontal parts:

$$d = dx^\lambda \otimes \partial_\lambda + dy^i \otimes \partial_i + dy^i_\mu \otimes \partial_i^\mu = [dx^\lambda \otimes (\partial_\lambda + y^i_\lambda\partial_i + y^i_{\lambda\mu}\partial_i^\mu)]$$
$$+ [(dy^i - y^i_\lambda dx^\mu) \otimes \partial_i + (dy^i_\mu - y^i_{\lambda\mu}dx^\lambda) \otimes \partial_i^\mu]$$
$$= \theta_1 + \theta_2 = d_H + d_V. \tag{0.38}$$

Let us consider the repeated jet manifold

$$J^1J^1E \to B$$

provided with the local coordinates

$$(x^\lambda, y^i, y^i_\lambda, y^i_{0\mu}, y^i_{\lambda\mu}).$$

There are two bundle morphisms over J^1E:

$$\pi_{1(01)}\colon J^1J^1E \to J^1E, \qquad y^i_\lambda \circ \pi_{1(01)} = y^i_\lambda,$$
$$j^1\pi_{01}\colon J^1J^1E \to J^1E, \qquad y^i_\lambda \circ j^1\pi_{01} = y^i_{0\lambda}.$$

In accordance with the affine structure of the bundle E^{01} (0.29), difference of these morphisms over J^1E yields the bundle morphism

$$j^1\pi_{01} - \pi_{1(01)} = \Delta\colon J^1J^1E \to T^*B \underset{E}{\otimes} VE,$$
$$(x^\lambda, y^i, \dot{x}_\lambda \otimes \dot{y}^i) \circ \Delta = (x^\lambda, y^i, y^i_{0\lambda} - y^i_\lambda).$$

The kernel of Δ is the affine subbundle $\widehat{J}^2E \subset J^1J^1E$ over J^1E which is characterized by the coordinate condition

$$y^i_{0\lambda} = y^i_\lambda.$$

The adapted coordinates on $\widehat{J}^2E$ are $(x^\lambda, y^i, y^i_\lambda, y^i_{\lambda\mu})$ where, in contrast to the coordinates on J^2E,

$$y^i_{\lambda\mu} \neq y^i_{\mu\lambda}.$$

Hence, there exist the following affine bundle monomorphisms over J^1E:

$$J^2E \to \widehat{J}^2E \to J^1J^1E, \tag{0.39}$$

which result in the canonical splitting of $\widehat{J}^2E$ over J^1E:

$$\widehat{J}^2E = J^2E \underset{J^1E}{\oplus} \left(\overset{2}{\wedge} T^*B \underset{E}{\otimes} VE\right).$$

0.3 General Connections.

In general, a connection on a bundle E must determine a certain lift of a tangent vector to the base B at a point $x \in B$ to tangent vectors to E at each point $y \in E$ projected to x. In other words, a connection Γ on a bundle E can be regarded as a certain morphism

$$\Gamma\colon E \underset{B}{\times} TB \to TE \tag{0.40}$$

which is both a linear bundle morphism

$$E \underset{B}{\times} TB \underset{E}{\to} TE$$

over E and a bundle morphism

$$E \underset{B}{\times} TB \underset{TB}{\to} TE \tag{0.41}$$

over TB.

One can introduce a connection in various equivalent ways. In the framework of the jet formalism, we do it as follows.

Given a bundle E, a connection Γ on E is defined to be a global section

$$\Gamma\colon E \to J^1E$$

of the bundle E^{01} (0.29). Its coordinate expression is

$$(x^\lambda, y^i, y^i_\lambda) \circ \Gamma = (x^\lambda, y^i, \Gamma^i_\lambda(y)).$$

Let Γ be a connection on a bundle E and

$$\Phi\colon E \to E'$$

be a bundle isomorphism. Then,

$$\Gamma' = j^1\Phi \circ \Gamma \circ \Phi^{-1}\colon E' \to J^1E'$$

is a connection on the bundle E'. In particular, if Φ is a bundle isomorphism over B, we have the coordinate expression

$$(x^\lambda, y'^i, y'^i_\lambda) \circ \Gamma' = (x^\lambda, \Phi^i, (\partial_\lambda\Phi^i + \Gamma^j_\lambda\partial_j\Phi^i) \circ \Phi^{-1}).$$

By means of the contact map θ_1 (0.31), a connection Γ can be represented by the projectable tangent-valued horizontal form

$$\theta_1 \circ \Gamma\colon E \to T^*B \underset{B}{\otimes} TE.$$

We denote this form by the same symbol Γ. It has the coordinate expression

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda(y)\partial_i). \tag{0.42}$$

The form Γ determines morphism (0.40):

$$\Gamma\colon (y, \partial_\lambda) \to (\partial_\lambda + \Gamma^i_\lambda(y)\partial_i) \in T_yE.$$

The horizontal lift

$$HE = TB \rfloor \Gamma(E) \subset TE$$

forms a distribution of horizontal subspaces of TE. This distribution obeys the equation

$$HE \rfloor \widehat{\Gamma} = 0$$

where $\widehat{\Gamma}$ is the vertical tangent-valued form on E defined by means of the contact form θ_2 (0.32):

$$\widehat{\Gamma} = \theta_2 \circ \Gamma\colon\ E \to T^*E \otimes VE,$$
$$\widehat{\Gamma} = (dy^i - \Gamma^i_\lambda(y)dx^\lambda) \otimes \partial_i. \tag{0.43}$$

The form (0.43) implies splitting of exact sequence (0.10).

Given a connection Γ, morphism (0.40) yields splitting of the exact sequence (0.9) and, as a consequence, the horizontal splitting of the tangent bundle

$$TE = HE \underset{E}{\oplus} VE.$$

Its coordinate expression

$$\dot x^\lambda \partial_\lambda + \dot y^i \partial_i = \dot x^\lambda(\partial_\lambda + \Gamma^i_\lambda(y)\partial_i) \oplus (\dot y^i - \dot x^\lambda \Gamma^i_\lambda(y))\partial_i$$

is derived from the splitting (0.34) by substituting a connection Γ:

$$y^i_\lambda = \Gamma^i_\lambda(y).$$

A connection Γ defines the bundle morphism

$$D\colon\ J^1E \ni w \to w - \Gamma(\pi_{01}(w)) \in T^*B \underset{E}{\otimes} VE$$

of the affine bundle E^{01} into the vector bundle

$$T^*B \underset{E}{\otimes} VE \to E.$$

We call this morphism a covariant differential. Its coordinate expression reads

$$D = (y^i_\lambda - \Gamma^i_\lambda(y))dx^\lambda \otimes \partial_i. \tag{0.44}$$

To describe totality of general connections on a bundle E, one can use the following fact.

Proposition. Let Γ be a connection and σ be some soldering form (0.22) on a bundle E. Then, their affine sum

$$\Gamma' = \Gamma + \sigma\colon\ E \to J^1E,$$
$$\Gamma' = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda(y)\partial_i + \sigma^i_\lambda(y)\partial_i),$$

over E is a connection on a bundle E. Let Γ and Γ' be connections on E. Then, their affine difference over E is a soldering form

$$\sigma = \Gamma - \Gamma'\colon\ E \to T^*B \underset{E}{\otimes} VE.$$

This general approach to connections is suitable to formulate the conventional concept of a principal connection. It is a connection A on a G-principal bundle P which obeys certain symmetries under the action of a structure group G on P and J^1P. Namely, a principal connection is a section of the bundle P^{01} which represents a G-equivariant bundle morphism such that the following diagram is commutative for each canonical morphism (0.4) and its jet prolongation:

$$\begin{array}{ccc} P & \xrightarrow{A} & J^1P \\ {\scriptstyle r_g}\downarrow & & \downarrow{\scriptstyle j^1 r_g} \\ P & \xrightarrow{A} & J^1P \end{array}$$

Given a principal bundle atlas Ψ^P and the associated canonical coordinates (0.17) on the bundle P, we have

$$A = dx^\lambda \otimes (\partial_\lambda + A^m_\lambda(p)\partial_m),$$
$$\widehat{A} = (dp^m - A^m_\lambda(p)dx^\lambda)\partial_m,$$
$$A^m_\lambda(x^\lambda, p^m(p))\partial_m = (r_g)_*(A^m_\lambda(x^\lambda, 0)\partial_m)$$
$$= A^m_\lambda(x^\lambda, 0)\,\mathrm{ad}\, g^{-1}(\partial_m), \quad p^m = a^m(g).$$

Using the trivial vertical splitting (0.16) of the vertical bundle VP, we can reproduce a familiar principal connection form

$$\overline{A} = \alpha \circ \widehat{A} = (dp^m - A^m_\lambda(p)dx^\lambda)J_m. \tag{0.45}$$

Note that, in the case of a principal bundle P, the exact sequence (0.9) entails the exact sequence

$$0 \to V^GP \to T^GP \to TB \to 0$$

where

$$V^GP = VP/G, \qquad T^GP = TP/G$$

denote the quotients of the bundles VP and TP by canonical action (0.4) of G on P and its tangent prolongation $(r_g)_*$ on TP. A principal connection A defines splitting of this sequence.

Let E be a bundle associated with a G-principal bundle P. A principal connection A on P induces an associated principal connection on E. The corresponding horizontal splitting of the tangent bundle TE is given by the relation

$$TE = (TP \times TF)/G_* = [(HP \oplus VP) \times TF]/G_* = HE \oplus VE.$$

With respect to associated atlases Ψ^P of P and Ψ of E, an associated principal connection A on E takes the coordinate form

$$A^i_\lambda(y) = A^m_\lambda(x) I_m(y^i), \qquad A^m_\lambda(x) = A^m_\lambda(x^\lambda, 0). \tag{0.46}$$

By $I_m(y)$, we here denote generators of the group G acting on a standard fibre F of the bundle E on the left (Section 1.1).

Proposition. Every connection Γ on a bundle E associated with some principal bundle P is represented by the affine sum

$$\Gamma = A + \sigma$$

of a principal connection A and a soldering form σ.

Let E be a vector bundle. A linear connection on E is a section Γ of the bundle E^{01} which defines linear bundle morphism (0.41). Its coordinate expression is

$$\Gamma^i_\lambda(y) = \Gamma^i_{\lambda j}(x) y^j = y^j \partial_j \Gamma^i_\lambda(y).$$

Every linear connection is obviously a principal connection.

Let E be an affine bundle modelled on a vector bundle $\overline{E}$. An affine connection on E is a section Γ of the bundle E^{01} which yields affine bundle morphism (0.41). It is given by the coordinate expression

$$\Gamma^i_\lambda(y) = a^i_\lambda(x) + \Gamma^i_{\lambda j}(x) y^j. \tag{0.47}$$

The fibred derivative $\overline{\Gamma}$ (0.8) of the section Γ is a linear morphism

$$\overline{\Gamma}: \overline{E} \to J^1\overline{E}$$

which defines some linear connection on the vector bundle $\overline{E}$. Its coordinate expression reads

$$\overline{\Gamma}^i_\lambda(\overline{y}) = \Gamma^i_{\lambda j}(x)\overline{y}^j.$$

Note that, in comparison with the linear connection $\Gamma^i_{\lambda j}(x)\overline{y}^j$, the term $\Gamma^i_{\lambda j}(x) y^j$ in expression (0.47) is not a linear connection since y are elements of an affine space which fails to admit linear transformations in general.

Let us consider the particular case of a vector bundle E provided with the canonical structure of an affine bundle. This affine bundle is associated with a principal bundle, and every linear connection $\overline{\Gamma}$ on E is uniquely extended to an affine principal connection on E. Hence, every affine connection Γ on a vector bundle E can be written as the sum

$$\Gamma = \overline{\Gamma} + \sigma \tag{0.48}$$

of a linear connection $\overline{\Gamma}$ and some basic soldering form σ.

We cite the basic differential operators involving a connection Γ represented by the form (0.42) and a soldering form σ:

(i) covariant differential (0.44) of a section e:

$$De = Dj^1 e = (\partial_\lambda e^i - \Gamma^i_\lambda \circ e) dx^\lambda \otimes \partial_i, \tag{0.49}$$

(ii) Nijenhuis differential d_Γ (0.23),

(iii) codifferential δ_Γ (0.27),

(iv) curvature (0.25):

$$K = \frac{1}{2} d_\Gamma \Gamma, \tag{0.50}$$

(v) torsion

$$\Omega = d_\sigma \Gamma = d_\Gamma \sigma, \tag{0.51}$$

(vi) cotorsion

$$W = \delta_\Gamma \sigma,$$

(vii) soldering curvature

$$\varepsilon = \frac{1}{2} d_\sigma \sigma,$$

(viii) Ricci tensor

$$R = \delta_\sigma K.$$

We have the following identities:

(i) the first Bianchi identity:

$$d_\Gamma \Omega = d^2_\Gamma \sigma = [K, \sigma] = -d_\sigma K,$$

(ii) the second Bianchi identity (0.26):

$$d_\Gamma K = 0,$$

(iii) if $\Gamma' = \Gamma + \sigma$, then

$$K' = K + \varepsilon + \Omega, \qquad \Omega' = \Omega + 2\varepsilon.$$

Chapter 1

CLASSICAL GAUGE THEORY

Differential geometry and jet bundle formalism provide us with the adequate mathematical formulation of classical gauge theory [DAN, EGU, IVA 1983, TRA 1984, MAR]. This formulation is based on the following propositions.

(i) Matter fields ϕ are represented by global sections of a vector bundle E associated with a principal bundle P with a structure Lie group G ($\dim G > 0$).

(ii) Principal connections on P are identified with gauge potentials which are mediators of interaction between matter fields ϕ.

(iii) A configuration space of fields ϕ is the jet manifold J^1E, and their momentum space is the Legendre bundle Π over E.

(iv) A Lagrangian L of fields ϕ on J^1E and their multimomentum Hamiltonian form H on Π are required to be gauge invariant.

(v) In the case of spontaneous symmetry breaking with an exact symmetry subgroup $H \subset G$, a global section of the quotient bundle P/H is treated as a classical Higgs field.

By X, we further denote a world manifold which is assumed to be 4-dimensional, oriented, and connected.

1.1 Geometric Theory of Classical Fields

In this Section, matter fields and gauge potentials are described by sections of a matter bundle E and a connection bundle C respectively. Configuration spaces of these fields are jet manifolds J^1E and J^1C on which the first order Lagrangian formalism is constructed.

Let us examine matter fields ϕ identified with global sections of a vector bundle

$$(E, \pi, X, F, G)$$

over a world manifold X. This bundle is assumed to be associated with a principal bundle

$$\pi_P\colon\ P \to X$$

where the structure group G acts on the standard fibre F of E on the left. We call E a matter bundle.

In field theory, an atlas $\Psi = \{U_\kappa, \psi_\kappa\}$ of the bundle E determines some reference frame in the sense that, with respect to Ψ, a section ϕ of E can be represented by a family of F-valued field functions

$$\phi_\kappa(x) = \psi_\kappa(x)\phi(x) = \phi^i(x)v_i, \qquad x \in U_\kappa,$$
$$\phi(x) = \phi^i(x)v_i(x).$$

Here, $\{v_i\}$ is a fixed basis for the standard fibre F and

$$v_i(x) = \psi_\kappa^{-1}(x)v_i$$

are bases (0.14) for fibres E_x which are called the bases associated with the atlas Ψ. For instance, we have

$$\phi_\kappa(x) = f_\phi(z_\kappa(x))$$

where $\{z_\kappa(x)\}$ is the associated atlas of the principal bundle P and f_ϕ is the F-valued equivariant function on P corresponding to ϕ.

Given coordinates (v^i) on the standard fibre F of E and a bundle atlas Ψ, the vector bundle E is provided with linear bundle coordinates (x^λ, y^i) (0.2) associated with Ψ:

$$y^i(y) = v^i \circ \psi_\kappa(y) \qquad y = y^i(y)v_i(\pi(y)). \tag{1.1}$$

These coordinates are adapted to the canonical vertical splitting (0.15) of the vertical bundle VE because of the identification

$$\partial_i = v_i(x).$$

Being endowed with this splitting, the vertical bundle VE is associated with the principal bundle P.

Let TX be the tangent bundle over a world manifold X. An atlas

$$\Psi^T = (U_\kappa, \psi_\kappa^T)$$

of the tangent bundle TX determines a world reference frame so that, given a fixed basis $\{t_a\}$ for the standard fibre $T = \mathbb{R}^4$ of TX, the frame

$$\{t_a(x)\} = (\psi_\kappa^T(x))^{-1}\{t_a\}, \qquad x \in U_\kappa, \tag{1.2}$$

associated with the atlas Ψ^T is erected at a point $x \in X$.

The structure group of the tangent bundle TX is

$$GL_4 = GL^+(4, \mathbb{R}).$$

The principal bundle LX associated with TX is isomorphic to the bundle of linear frames, i.e., ordered bases for tangent spaces T_xX. Namely, one identifies an element $g \in GL_4$ with the frame $\{g_a^b t_b\}$ where g_a^b is the matrix representation of g on T. Given an atlas Ψ^T, associated frame functions (1.2) then represent local sections $z_\kappa^T(x)$ of the principal bundle LX which are associated with the atlas Ψ^T. By

$$\Psi^T = \left\{U_\kappa, \psi_\kappa^T(x) = \left[z_\kappa^T(x)\right]_T^{-1}\right\},$$

we therefore denote atlases both of tensor bundles and of LX.

As distinct from other bundles over X with the standard fibre $\mathbb{R}^4$, atlases of the tangent bundle are equivalent to a holonomic atlas

$$\Psi^T = \left\{U_\kappa, \psi_\kappa^T = (\chi_\kappa)_*\right\}$$

correlating with some coordinate atlas

$$\Psi_X = \{U_\kappa, \chi_\kappa\}$$

of the base manifold X. The associated holonomic basis vectors

$$t_\mu(x) = \partial_\mu$$

are tangent to coordinate curves in X. The associated bases for cotangent spaces T_x^*X are $\{dx^\mu\}$. The associated bundle coordinates on TX are the induced coordinates $(x^\mu, \dot{x}^\mu)$.

For the sake of simplicity, we further choose an open covering $\{U_\kappa\}$ of X which is the same for bundle atlases Ψ, Ψ^T and coordinate atlases Ψ_X of X.

Given an atlas $\{z_\kappa\}$, the principal bundle P is endowed with the canonical bundle coordinates (x^μ, p^m) (0.17). For instance, let f_ϕ be an equivariant function on P corresponding to a section ϕ of the associated bundle E. In the coordinates (0.17), we have

$$p^m(z_\kappa(x)) = a^m(1_G) = 0,$$
$$\phi_\kappa(x) = f_\phi(z_\kappa(x)) = f_\phi(x^\mu, 0).$$

Let A be a principal connection on the principal bundle P and $\overline{A}$ be its connection form (0.45). Given the associated principal connection A on E, the covariant derivative

$$D_\tau\phi = \tau\rfloor D\phi\colon\ X \to VE$$

of a section ϕ along a vector field τ on X represents a section of the vertical bundle VE. Because of the canonical vertical splitting (0.15) of VE and the canonical isomorphism $TF = F$, this section is determined by the F-valued equivariant function

$$\tau^H\rfloor(f_\phi).$$

on P where

$$\tau^H(p) = \tau(\pi_P(p))\rfloor A(p) \in \mathcal{P}(P)$$

is the horizontal lift of a vector field τ on X. Given associated atlases Ψ^P and Ψ, we have the following coordinate expressions:

$$\begin{aligned}
\tau &= \tau^\mu(x)\partial_\mu,\\
\tau^H &= \tau^\mu(x)(\partial_\mu + A_\mu^m(p)\partial_m),\\
D_\tau\phi_\kappa(x) &= \tau^\mu(x)D_\mu\phi_\kappa(x),\\
D_\mu\phi_\kappa(x) &= (\partial_\mu^H f_\phi)(x^\mu, 0) = (\partial_\mu + A_\mu^m(x)\partial_m)f_\phi(x^\mu, 0)\\
&= (\partial_\mu - A_\mu^m(x)I_m)\phi_\kappa(x).
\end{aligned}$$

For the sake of simplicity, we here use the formal relation

$$\partial_m f_\phi(x^\mu, p^m) = \partial_m(\exp[-p^m I_m])f_\phi(x^\mu, 0) = -I_m f_\phi(x^\mu, p^m).$$

The covariant differential (0.49) of field functions ϕ_κ hence reads

$$D\phi_\kappa = dx^\mu \otimes (\partial_\mu - A_\mu^m(x)I_m)\phi_\kappa(x)$$

where

$$A_\kappa = A_\mu^m(x)I_m dx^\mu = (z_\kappa)^*\overline{A} \tag{1.3}$$

is the local connection 1-form. Its coefficients $A_\mu^m(x)$ are the coefficients of the associated principal connection (0.46) on E:

$$A_\mu^i(y) = A_\mu^m(x)I_m{}^i{}_j y^j,$$
$$D = (y_\mu^i - A_\mu^m(x)I_m{}^i{}_j y^j)dx^\mu \otimes v_i(x)$$

where, given the coordinates (1.1) on the bundle E, (x^μ, y^i, y_μ^i) are the adapted coordinates on the jet manifold J^1E.

Remark. Since $\mathfrak{g}$ is the left Lie algebra, generators $I_m(y)$ of the group G acting on the standard fibre F on the left coincide with $-J_m(y)$ where J_m is a basis for $\mathfrak{g}$.

Let us recall that principal connections on a principal bundle P with a structure group G are represented by G-equivariant sections

$$P \to J^1P$$

of the jet bundle P^{01}. As a consequence, there is 1:1 correspondence between principal connections A on P and global sections A^C of the affine bundle

$$C = P^{01}/G = (J^1P/G \to P/G = X) \tag{1.4}$$

modelled on the vector bundle

$$\overline{C} = T^*X \otimes V^G P. \tag{1.5}$$

We call C the connection bundle.

Remark. Sections of the bundle $V^G P$ are vertical vector fields on P invariant under the canonical action of G on P on the right. This bundle is associated with the principal bundle P. Its standard fibre is the right Lie algebra $\widetilde{\mathfrak{g}}$ of right-invariant vector fields on G. The structure group G acts on this standard fibre by the adjoint representation. Fibres of the bundle $V^G P$ therefore fail to be canonically isomorphic to the standard fibre in general. Global sections of the bundle $V^G P$ form the infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}(X)$ called the gauge Lie algebra.

Given an atlas $\{z_\kappa\}$ of P, the bundle $V^G P$ is provided with the associated bundle coordinates (x^μ, k^m) such that right-invariant vertical vector fields

$$u(p) = \dot{p}^m(z_\kappa(x)g)\partial_m = \dot{p}^m(z_\kappa(x))\mathrm{ad}\, g^{-1}(\partial_m)$$

on P are represented by sections of the bundle $V^G P$:

$$u^G(x) = k^m(x)I_m = \dot{p}^m(z_\kappa(x))I_m$$

where $\{I_m\}$ is a basis for the right Lie algebra $\widetilde{\mathfrak{g}}$. The corresponding bundle coordinates on C are (x^μ, k^m_μ). A section A^C of the bundle C then has the coordinate expression

$$(k^m_\mu \circ A^C)(x) = A^m_\mu(x)$$

where $A^m_\mu(x)$ are coefficients of local connection 1-form (1.3). In gauge theory, sections A^C are treated as gauge potentials.

Remark. Recalling the contact map (0.31), we may represent a section of the affine bundle (1.4) as the form (0.42):

$$A^C\colon X \to T^*X \otimes T^G P,$$
$$A^C = dx^\mu \otimes (\partial_\mu - A^m_\mu(x)I_m).$$

A finite-dimensional configuration space of matter fields ϕ is the jet manifold J^1E.

A configuration space of gauge potentials is the jet manifold J^1C. The affine bundle C^{01} admits the canonical splitting

$$J^1C = C_+ \underset{C}{\oplus} C_- = (J^2P/G) \underset{C}{\oplus} (\overset{2}{\wedge} T^*X \otimes V^G P) \tag{1.6}$$

where C_+ is the affine bundle modelled on the vector bundle

$$\overline{C}_+ = \overset{2}{\vee} T^*X \otimes V^G P \tag{1.7}$$

[MAN 1985]. Local coordinates

$$(x^\mu, k^m_\mu, s^m_{\mu\lambda}, F^m_{\lambda\mu}) = (x^\mu, k^m_\mu, k^m_{\mu\lambda} + k^m_{\lambda\mu}, k^m_{\mu\lambda} - k^m_{\lambda\mu} - c^m_{nl}k^n_\lambda k^l_\mu) \tag{1.8}$$

on J^1C are adapted both to the submanifold C_+ and to the submanifold

$$C_- = C \underset{X}{\times} (\overset{2}{\wedge} T^*X \otimes V^G P).$$

Here, c^m_{nl} are the structure constants of the group G.

Remark. To get the splitting (1.6), one can use the monomorphisms (0.39) and the canonical isomorphism of $\widehat{J}^2P/G$ to J^1C.

Projection pr_2 of the splitting (1.6) defines the fundamental form

$$\mathcal{F}\colon J^1C \to \overset{2}{\wedge} T^*X \otimes V^G P,$$
$$\mathcal{F} = \frac{1}{2}(k^m_{\mu\lambda} - k^m_{\lambda\mu} - c^m_{nl}k^n_\lambda k^l_\mu)dx^\lambda \wedge dx^\mu \otimes I_m. \tag{1.9}$$

For instance, if A is a principal connection on P, its curvature (0.50) is given by the expression

$$\mathcal{F}_A = \mathcal{F} \circ j^1 A^C.$$

Note that, to construct gauge invariant Lagrangians of gauge potentials, one uses only the form (1.9), whereas the form

$$S\colon J^1C \to C_+$$

is defined by gauge conditions (Section 1.3).

Let us consider the first order Lagrangian formalism on the configuration spaces J^1E and J^1C of matter fields and gauge potentials. We further assume that X is a n-dimensional manifold endowed with a world metric g.

Remark. By a world metric, we call a fibre metric on a world manifold X.

A first order Lagrangian is defined to be a morphism

$$L\colon J^1E \to \overset{n}{\wedge} T^*X.$$

In the adapted coordinates $(x^\lambda, y^i, y^i_\lambda)$ on the jet manifold J^1E, a Lagrangian L is expressed as a horizontal form on J^1E:

$$L = \widetilde{\mathcal{L}}(x^\lambda, y^i, y^i_\lambda)\mathcal{V} = \mathcal{L}\omega \tag{1.10}$$

where $\widetilde{\mathcal{L}}$ is a real function on J^1E, $\mathcal{V}$ is volume the form (0.20) and

$$\omega = dx^{\lambda_1} \wedge \ldots \wedge dx^{\lambda_n}.$$

We call

$$\mathcal{L} = \widetilde{\mathcal{L}}\sqrt{|g|}$$

a Lagrangian density.

The following objects are usually associated with the Lagrangian (1.10).

(i) *The Legendre morphism*

One calls the linear bundle

$$\Pi = \overset{n}{\wedge} T^*X \otimes TX \underset{E}{\otimes} V^*E \tag{1.11}$$

over E the Legendre bundle. This is provided with the so-called standard bundle coordinates $(x^\lambda, y^i, p^\lambda_i)$ such that

$$\begin{aligned} &\pi \circ \pi_\Pi\colon \Pi \to E \to X, \\ &\pi \circ \pi_\Pi\colon (x^\lambda, y^i, p^\lambda_i) \to (x^\lambda, y^i) \to (x^\lambda). \end{aligned} \tag{1.12}$$

The Legendre morphism

$$\widehat{L}\colon J^1E \to \Pi$$

is defined to be the fibre derivative of L given by the coordinate expression

$$(x^\lambda, y^i, p^\lambda_i) \circ \widehat{L} = (x^\lambda, y^i, p^\lambda_i = \pi^\lambda_i(w) = \partial^\lambda_i \mathcal{L}). \tag{1.13}$$

So, the Legendre bundle (1.11) makes the sense of a finite-dimensional momentum space of fields.

(ii) *The Poincaré-Cartan form*

The Poincaré-Cartan form is defined to be

$$\begin{aligned} &\Theta = L + K\colon J^1E \to \overset{n}{\wedge} T^*E, \\ &K = \pi^\lambda_i(\theta_2 \rfloor dy^i) \wedge \omega_\lambda = \pi^\lambda_i dy^i \wedge \omega_\lambda - \pi^\lambda_i y^i_\lambda \omega, \\ &\Theta = \pi^\lambda_i dy^i \wedge \omega_\lambda - (\pi^\lambda_i y^i_\lambda - \mathcal{L})\omega \end{aligned} \tag{1.14}$$

where the form ω_λ is given by expression (0.19)

(iii) *The Euler-Lagrange operator*

The Euler-Lagrange operator is defined to be the bundle morphism

$$\begin{aligned} &\mathcal{E}(L)\colon J^2E \to \overset{n}{\wedge} T^*X \wedge V^*E, \\ &\mathcal{E}(L) = (\partial_i \mathcal{L} - d_\lambda \pi^\lambda_i) dy^i \wedge \omega = \delta_i \mathcal{L} dy^i \wedge \omega, \end{aligned} \tag{1.15}$$

where d_λ are the total derivatives (0.37) and operators

$$\delta_i = \partial_i - d_\lambda \partial^\lambda_i$$

are called the variation derivatives.

Given a Lagrangian L and the jet prolongation j^1e of a section e of the bundle E, one can define a familiar Lagrangian form on X:

$$\begin{aligned} &L(e)\colon X \to \overset{n}{\wedge} T^*X, \\ &L(e) = (j^1e)^*L = \mathcal{L}(x^\lambda, e^i(x), \partial_\lambda e^i(x))\omega. \end{aligned}$$

The corresponding action functional reads

$$S(e) = \int_N L(e)$$

where $N \subset X$ is a compact n-dimensional submanifold with (n-1)-dimensional boundary ∂N. A section e of the bundle E is called critical if, for any N, the action functional $S(e)$ is stationary at e. It takes place if and only if a section e satisfies the equations

$$(j^2e)^*[u \rfloor \mathcal{E}(L)] = 0 \tag{1.16}$$

for every vertical vector field u on E. In the adapted coordinates, these equations take the familiar form of the Euler-Lagrange equations

$$(j^2e)^*[\delta_i \mathcal{L}] = \frac{\partial \mathcal{L}(e)}{\partial e^i} - \partial_\lambda \frac{\partial \mathcal{L}(e)}{\partial e^i_\lambda} = 0. \tag{1.17}$$

Given a Lagrangian L, one can derive the Euler-Lagrange operator (1.15) from the canonical splitting of dL:

$$\begin{aligned} &dL = \mathcal{E} + \Delta, \\ &\mathcal{E} = \lambda^* dL, \quad \lambda^*(\Delta) = 0, \end{aligned} \tag{1.18}$$

where the morphism λ^* is defined as follows.

Let ε be a morphism

$$\begin{aligned} &\varepsilon\colon J^1E \to \overset{n}{\wedge} T^*X \otimes V^*J^1E, \\ &\varepsilon = \omega \otimes (\varepsilon_i dy^i + \varepsilon^\lambda_i dy^i_\lambda), \end{aligned}$$

where ε, ε_i^μ are local functions on J^1E. The morphism λ^* is expressed as

$$\lambda^*\varepsilon\colon\ J^2E \to \overset{n}{\wedge} T^*X \otimes V^*E,$$
$$\lambda^*\varepsilon = \omega \otimes (\varepsilon_i - d_\lambda \varepsilon_i^\lambda) dy^i.$$

For a Lagrangian L, we then have

$$dL\colon\ J^1E \to \overset{n}{\wedge} T^*X \wedge V^*J^1E,$$
$$dL = (\partial_i \mathcal{L} dy^i + \partial_i^\lambda \mathcal{L} dy_\lambda^i) \wedge \omega,$$
$$\lambda^* dL = \delta L = \mathcal{E}(L),$$

where we introduce the variation operator

$$\delta = \lambda^* d = dy^i \otimes \delta_i.$$

The Euler-Lagrange operator (1.15) is equal to zero if the Lagrangian (1.10) is the horizontal differential:

$$L = \mathrm{d}_H a = d_\lambda a^\lambda \omega$$

where the operator d_H is given by expression (0.38) and

$$a = a^\lambda(y)\omega_\lambda \tag{1.19}$$

is some horizontal (n–1)-form on E. We have

$$\mathcal{E}(L) = \delta \mathcal{L} = \delta \mathrm{d}_H a = 0.$$

Conversely, let us suppose that

$$\mathcal{E}(L) = 0.$$

Then, there exists local horizontal form (1.19) such that

$$L = \mathrm{d}_H a.$$

In field theory, Lagrangians are usually required to be gauge invariant. There are different types of gauge transformations. We discuss them in the next Section.

1.2 Gauge Transformations

We have assumed that a matter bundle E over a world manifold X is reduced to a bundle associated with a principal bundle P with some structure group G acting on the standard fibre F of E on the left. In this case, one may examine gauge transformations corresponding to the group G and may require Lagrangian (1.10) on J^1E be invariant under these transformations.

There are two main types of gauge transformations. These are atlas transformations and associated principal morphisms.

At first, we consider gauge transformations associated with internal symmetries which do not concern the tangent bundle TX over a base manifold X.

In field theory, atlas transformations are treated as transformations of reference frames. In virtue of the above-mentioned assumption, we restrict our consideration to the atlases $\Psi = \{U_\kappa, \psi_\kappa\}$ of the bundle E which are associated with atlases

$$\Psi^P = \{z_\kappa(x), x \in U_\kappa\} \tag{1.20}$$

of the principal bundle P. In this case, transition functions of Ψ are G-valued.

If an open covering $\{U_\kappa\}$ is fixed, atlas transformations

$$\Psi^P = \{z_\kappa(x)\} \to \Psi'^P = \{z'_\kappa(x) = z_\kappa(x) g_\kappa^{-1}(x)\}, \quad g_\kappa \in G(U_\kappa), \tag{1.21}$$

form the group

$$G(\{U_\kappa\}) = \prod G(U_\kappa)$$

where $G(U_\kappa)$ is the group of G-valued functions on U_κ. All atlas transformations form the group

$$G(X)_A = G(\{U_\kappa\}_{\max})$$

where $\{U_\kappa\}_{\max}$ is the covering for the maximal atlas of the principal bundle P. Transformations (1.21) do not alter sections $\phi(x)$ of the bundle E, but change their representation by field functions $\phi_\kappa(x)$:

$$\phi'_\kappa(x) = f_\phi(z'_\kappa(x)) = f_\phi(z_\kappa(x) g_\kappa^{-1}(x)) = g_\kappa(x) f_\phi(z_\kappa(x))$$
$$= g_\kappa(x)\phi_\kappa(x), \qquad x \in U_\kappa.$$

Associated principal morphisms Φ_E of the bundle E are the bundle morphisms (0.7) induced by principal isomorphisms (0.6) of the principal bundle P over the identity morphism of its base X. In contrast to atlas transformations, the morphisms Φ_E alter sections of E:

$$f_\phi(p) \to f_{\phi'}(p) = f_\phi(p f_s^{-1}(p)).$$

At the same time, given some atlas (1.20), isomorphism (0.6) defines the new atlas

$$\Psi'^P = \{z'_\kappa(x) = z_\kappa(x) f_s^{-1}(z_\kappa(x)), \quad x \in U_\kappa\}$$

with the same transition functions. As a consequence, given an associated atlas

$$\Psi = \{U_\kappa, \psi_\kappa\}$$

of E, there are the following local relations

$$(\psi_\kappa \circ \Phi_E \circ \phi)(x) = (f_s(z_\kappa(x)) \circ \psi_\kappa \circ \phi)(x), \quad x \in U_\kappa, \tag{1.22}$$

between the associated principal morphisms and atlas transformations. The gauge group $G(X)$ of principal isomorphisms differs from the group $G(X)_A$ of atlas transformations. For every bundle chart $(U_\kappa, \psi^P_\kappa)$, we however obtain

$$G(U_\kappa) = G(U_\kappa)_A. \tag{1.23}$$

The associated principal morphisms Φ_E have the jet prolongations $j^1\Phi_E$. One can require Lagrangian density (1.10) be invariant under the morphisms

$$j^1\Phi_E = j^1\widehat{P}_s, \qquad \widehat{P}_s = \widehat{P}_E|_{E\times s(X)},$$

for all global sections s of the principal group bundle $\widetilde{P}$. Moreover, we can replace global sections of the bundle $\widetilde{P}$ by its local sections and, in turn, by their jet prolongations. In other words, let us require a Lagrangian density $\mathcal{L}$ be invariant under the jet prolongations

$$j^1\widehat{P}_E\colon J^1\widetilde{P} \times J^1E \to J^1E$$

of the canonical morphisms $\widehat{P}_E$ (0.5), that is,

$$\mathcal{L}(w) = \mathcal{L}(j^1\widehat{P}_E(q, w)) \tag{1.24}$$

for all elements $w \in J^1E$ and $q \in J^1\widetilde{P}$.

This gauge invariance condition is sufficient for Lagrangian (1.10) to be invariant under the associated principal morphisms.

On the other hand, the gauge invariance condition (1.24) is equivalent to the invariance of a Lagrangian density $\mathcal{L}$ under atlas transformations. Indeed, let (U_κ, ψ_κ) be a bundle chart. In virtue of the relation (1.22) and the isomorphism (1.23), there exists a gauge transformation g_κ so that

$$f_s(z_\kappa(x)) = g_\kappa(x), \qquad x \in U_\kappa,$$
$$j^1(g_\kappa \circ \psi_\kappa)(w) = j^1\widehat{P}_s(w)$$

for every local section s of $\widetilde{P}$ on U_κ, and vice versa.

Comparing atlas transformations and principal morphisms, we may say that the former provide us with more opportunities for taking into account global topological characteristics of bundles. In particular, if a structure group of a bundle E is reducible to its subgroup H, one can always choose a bundle atlas with H-valued transition functions, whereas principal morphisms can not change transition functions of bundle atlases.

The necessary condition of the gauge invariance of Lagrangian L (1.10) consists in the fact that L must be brought into zero by generators of infinitesimal principal morphisms. These generators are associated with certain vertical vector fields on the bundle E as follows.

Let us consider principal isomorphisms $\Phi^l \in G(X)$ of P which form a flow

$$\Phi^l(p) = c(p, l),$$
$$\frac{dc}{dl} = u_P(c(p, l)), \quad c(p, 0) = p,$$

along some vertical vector field u_P on P. Note that, since isomorphisms Φ^l are G-equivariant, the field u_P can be identified with a global section of the bundle V^GP.

The principal isomorphisms Φ^l form a 1-parameter Lie subgroup of the gauge group $G(X)$ whose generator is a generator of the corresponding infinitesimal principal isomorphisms. The bundle V^GP is isomorphic to the quotient of the vertical bundle $V\widetilde{P}$ by the action of the group bundle $\widetilde{P}$ on itself on the right. It follows that there is 1:1 correspondence between generators of infinitesimal principal morphisms and right-invariant sections of the bundle $V\widetilde{P}$.

Remark. A suitable Sobolev completion of the gauge group $G(X)$ is a Banach Lie group. Its Lie algebra is a suitable Sobolev completion of the gauge Lie algebra $\widetilde{\mathfrak{g}}$ [MAR].

The principal isomorphisms Φ^l induce the associated principal morphisms (0.7) of the bundle E.

Given a point $y \in E$, let us consider the restriction of the canonical morphism $\widehat{P}_E$ to

$$\widehat{P}_y\colon \widetilde{P}_x \to \widehat{P}_E(y, \widetilde{P}_x) \subset E_x, \qquad \pi(y) = x,$$

and the corresponding tangent morphism

$$(\widehat{P}_y)_*\colon V\widetilde{P}_x \to VE_x, \qquad \pi(y) = x,$$
$$(\widehat{P}_y)_*\colon V_{1_x}\widetilde{P} \to V_yE,$$

where 1_x is the unit element of the group $\widetilde{P}_x$. Every local right-invariant section s_V of the vertical bundle $V\widetilde{P}$ induces a local vertical field on the bundle E:

$$u_{\mathfrak{p}}(y) = (\widehat{P}_y)_*(s_V(1_x)) \in V_yE, \qquad \pi(y) = x.$$

We call it a principal vertical vector field.

There is 1:1 correspondence between the local principal vector fields and the generators of local infinitesimal principal morphisms of the bundle E. We call them the gauge generators.

In order to define the gauge generators acting on Lagrangian (1.10), we can construct the lift (0.35) of principal vertical vector fields onto the jet manifold J^1E:

$$\overline{u}_{\mathfrak{p}} = u^i_{\mathfrak{p}}\partial_i + (\partial_\lambda u^i_{\mathfrak{p}} + y^j_\lambda \partial_j u^i_{\mathfrak{p}})\partial^\lambda_i.$$

The corresponding gauge generators act on $\mathbb{R}$-valued forms and tangent valued forms on J^1E as the Lie derivatives (0.24) given by the [F-N] brackets:

$$L_{\overline{u}_\mathfrak{g}}\phi = d_{\overline{u}_\mathfrak{g}}\phi = [\overline{u}_\mathfrak{g}, \phi]_{FN}.$$

In particular, the gauge generators act on Lagrangian L (1.10) by the rule:

$$\begin{aligned} L_{\overline{u}_\mathfrak{g}}(L) &= [\overline{u}_\mathfrak{g}, L]_{FN} \\ &= (u^i_\mathfrak{g}\partial_i + (\partial_\lambda u^i_\mathfrak{g} + y^j_\lambda \partial_j u^i_\mathfrak{g})\partial^\lambda_i)\mathcal{L}\omega. \end{aligned} \tag{1.25}$$

If a Lagrangian density $\mathcal{L}$ satisfies the gauge invariance condition (1.24), we have

$$L_{\overline{u}_\mathfrak{g}}\mathcal{L} = 0 \tag{1.26}$$

for all principal vertical vector fields $u_\mathfrak{g}$. This equality makes the sense of some conservation laws and provides us with certain conditions on the constitution of a gauge invariant Lagrangian.

The general approach to constructing gauge invariant Lagrangians consists in manipulating objects with linear gauge transformation laws:

(i) a vector bundle E;

(ii) the covariant differential (0.44)

$$D(w) \in T^*X \underset{E}{\otimes} VE, \qquad w \in J^1E,$$

and the integrable vertical splitting of VE;

(iii) a G-invariant fibre metric

$$a^E\colon X \to \overset{2}{\vee} E^*$$

in the bundle E.

With respect to the atlases Ψ of E associated with atlases (1.20), a fibre metric a^E however takes a canonical G-invariant form and, therefore, this is not a field quantity in general.

In the case of unbroken internal symmetries, a total Lagrangian L of gauge theory is defined on the configuration space

$$J^1E \underset{X}{\times} J^1C.$$

Let us denote the coordinates on this configuration space using the condensed notations

$$(x^\mu, q^A, q^A_\mu), \qquad q^A = (y^i, k^m_\mu)$$

and let us calculate Lie derivative (1.25) of a total Lagrangian L. As a consequence of the splitting (1.18), we obtain

$$L_{\overline{u}_\mathfrak{g}}L = [u^A_\mathfrak{g}\delta_A\mathcal{L} + d_\lambda(u^A_\mathfrak{g}\partial^\lambda_A\mathcal{L})]\omega = 0. \tag{1.27}$$

A local principal vertical field $u_\mathfrak{g}$ on the bundle $E \times_X C$ takes the coordinate form

$$\begin{aligned} u_\mathfrak{g} &= [u^A_m(q^B)\alpha^m(x^\mu) + u^{A\lambda}_m(q^B)\partial_\lambda\alpha^m(x^\mu)]\partial_A \\ &= \alpha^m(x^\mu)I_m{}^i{}_j y^j\partial_i + (\partial_\lambda\alpha^m(x^\mu) + c^m_{nl}k^n_\lambda\alpha^n(x^\mu))\partial^\lambda_m \end{aligned} \tag{1.28}$$

where $\alpha^m(x^\mu)$ are arbitrary local functions on X. Substituting this expression into equality (1.27), one reproduces the familiar Noëther identities for a gauge invariant Lagrangian:

$$\begin{aligned} & u^A_m\delta_A\mathcal{L} + d_\mu(u^A_m\partial^\mu_A\mathcal{L}) = 0, \\ & u^{A\lambda}_m\delta_A\mathcal{L} + d_\mu(u^{A\lambda}_m\partial^\mu_A\mathcal{L}) + u^A_m\partial^\lambda_A\mathcal{L} = 0, \\ & u^{A\lambda}_m\partial^\mu_A\mathcal{L} + u^{A\mu}_m\partial^\lambda_A\mathcal{L} = 0. \end{aligned}$$

Let q be some section of the bundle $E \times_X C$. Let us rewrite the conservation law (1.27) in the form

$$(j^2q)^*[u^A_\mathfrak{g}\delta_A\mathcal{L}] = (j^2q)^*[d_\lambda(u^A_\mathfrak{g}\partial^\lambda_A\mathcal{L})]$$

and integrate this equality over some compact submanifold $N \subset X$ with 3-dimensional boundary ∂N. Since functions $\alpha(x)$ in expression (1.28) are arbitrary, we can choose

$$\alpha^m(x) = 0, \qquad \partial_\mu\alpha^m(x) = 0, \qquad x \in \partial N.$$

In this case, the integral equation derived from equality (1.27) results in the relation

$$(j^2q)^*[u^A_m\delta_A\mathcal{L} - d_\lambda(u^{A\lambda}_m\delta_A\mathcal{L})] = 0. \tag{1.29}$$

This relation makes the sense of the constraint on the variation derivatives

$$(j^2q)^*[\delta_A\mathcal{L}]$$

which therefore fail to be independent.

A total Lagrangian of gauge theory is the sum

$$L = L_{(m)} + L_{(A)}$$

of a matter field Lagrangian $L_{(m)}$ and a Lagrangian $L_{(A)}$ of gauge potentials.

We here give an example of scalar matter fields possessing only internal symmetries. Let a^E be a G-invariant metric in the standard fibre F of a matter bundle and Γ be a connection on E. The familiar scalar field Lagrangian $L_{(m)}$ and the corresponding Euler-Lagrange operator read

$$L_{(m)} = \frac{1}{2}[g^{\mu\nu}a^E_{ij}(y^i_\mu - \Gamma^i_\mu(y))(y^j_\nu - \Gamma^j_\nu(y)) - m^2 a^E_{ij}y^i y^j]\sqrt{|g|}\,\omega,$$
$$\mathcal{E}(L) = -a^E_{ik}[m^2 y^i + g^{\mu\nu}(y^i_{\mu\nu} - y^j_\mu \partial_j \Gamma^i_\nu(y))]\sqrt{|g|}\,\omega \otimes dy^k,$$
$$\Gamma^i_\mu(y) = k^m_\mu I_m{}^i{}_j y^j. \tag{1.30}$$

Remark. For the sake of simplicity, we do not include the conformal term

$$-\frac{1}{6}a^E_{ij}y^i y^j R$$

where R is the scalar curvature associated with a world metric g.

The conventional Yang-Mills Lagrangian $L_{(A)}$ of gauge potentials on the jet manifold J^1C provided with the coordinates (1.8) is given by the expression

$$L_A = \frac{1}{4\varepsilon^2}a^G_{mn}g^{\lambda\mu}g^{\beta\nu}\mathcal{F}^m_{\lambda\beta}\mathcal{F}^n_{\mu\nu}\sqrt{|g|}\,\omega \tag{1.31}$$

where a^G is an adjoint-invariant metric in the Lie algebra $\tilde{\mathfrak{g}}$ and ε^2 is a coupling constant.

Remark. If G is semisimple,

$$a^G_{mn} = c^b_{mk}c^k_{nb}.$$

If G is compact, there is a basis $\{I_m\}$ for $\tilde{\mathfrak{g}}$ such that

$$a^G_{mn} = -2\delta_{mn}.$$

Now, let us consider world transformations.

Transformations of atlases of the tangent bundle TX form the group $GL_4(X)_A$ which contains the subgroup of holonomic transformations accompanied by the corresponding transformations of coordinate atlases:

$$\partial_\mu \to \frac{\partial x'^\nu}{\partial x^\mu}\partial'_\nu,$$
$$x^\mu = \chi^\mu(x) \to x'^\mu = \chi'^\mu(x). \tag{1.32}$$

Though nonholonomic atlas transformations are admitted, the holonomic reference frames are preferable in a sense. In contrast with other bundles over X, elements of the tangent bundle TX play the role of the operators of derivatives, and sections of TX form the sheaf of Lie algebras $\mathcal{T}(X)$. As distinct from the holonomic basis vectors ∂_λ, the components ∂_a of a nonholonomic frame fail to commute with each other and so, they can not be considered as elementary objects. For example, one may provide the jet manifold J^1E of a bundle E with the local coordinates (x^μ, y^i, y^i_a) associated with a nonholonomic atlas of TX:

$$y^i_a(y) = y^i_a(j^1_x e) = \partial_a e^i.$$

On the 2-jet manifold J^2E, we however can not define such coordinates in the intrinsic way because $y^i_{\lambda\mu} = y^i_{\mu\lambda}$, but

$$\partial_a\partial_b e^i \neq \partial_b\partial_a e^i.$$

To construct Lagrangian (1.10), we need a tetrad field h (Section 2.2) or a world metric g. In contrast with a fibre metric a^E in a bundle E, metric functions of g have no a GL_4-invariant form and take a canonical form only with respect to nonholonomic atlases of TX in general. It follows that, in gauge theory of world symmetries, a world metric g is a dynamic variable and a total Lagrangian is invariant under atlas transformations (1.32) if it is represented by the scalar-valued form (1.10).

In view of the canonical algebraic structure of the sheaf $\mathcal{T}(X)$, bundle morphisms of TX must be restricted to those which yield the isomorphisms $(\Phi_X)_*$ of TX tangent to diffeomorphisms Φ_X of X.

In the combined case of internal and world symmetries, one can consider the group $\mathrm{Diff}_X P$ of general principal isomorphisms

$$\Phi_P(pg) = \Phi_P(p)g, \qquad p \in P,\ g \in G. \tag{1.33}$$

of a principal bundle P. They are projected to diffeomorphisms Φ_X of a world manifold X and are accompanied by the tangent isomorphisms $(\Phi_X)_*$ of tensor bundles.

Remark. There can exist diffeomorphisms of X which fail to be projections of general principal isomorphisms of P.

General principal isomorphisms of P (1.33) induce associated general principal morphisms Φ_E of an associated bundle E by rule (0.7).

Let Φ^t be general principal isomorphism (1.33) of the principal bundle P which form a flow along some nonvertical vector field u_P on P projected to a vector field u_X on X. The isomorphism Φ^t induce the general principal morphisms Φ^t_E of the associated bundle E which form a flow along some projectable vector field u (0.11) on E projected to the same vector field u_X on X. We call u a principal vector field. By analogy with principal vertical vector fields, we can associate a principal vector field with a generator of some infinitesimal general gauge transformation.

In order to define such a generator acting on Lagrangian (1.10), we construct the lift (0.35) of a principal vector field u on the bundle E onto the jet manifold J^1E:

$$\overline{u} = u^\lambda\partial_\lambda + u^i\partial_i + (\partial_\lambda u^i + y^i_\lambda\partial_j u^i - y^i_\mu\partial_\lambda u^\mu)\partial^\lambda_i.$$

To calculate the Lie derivative $L_{\overline{u}}(L)$, let us single out the vertical component of the field $\overline{u}$. We can use the morphisms θ_2 and θ_2 given by coordinate expression (0.36). They yield the canonical decomposition.

$$\overline{u} = \overline{u}_V + \overline{u}_H \tag{1.34}$$

of the vector field $\overline{u}$ into the vertical (VJ^1E)-valued vector field $\overline{u}_V$ and the horizontal part $\overline{u}_H$:

$$\begin{aligned}\overline{u}_V &= (u^i - y^i_\lambda u^\lambda)\partial_i + (\partial_\lambda u^i + y^j_\lambda\partial_j u^i - y^i_\mu\partial_\lambda u^\mu - y^i_{\lambda\mu}u^\mu)\partial^\lambda_i,\\ \overline{u}_H &= u^\lambda(\partial_\lambda + y^i_\lambda\partial_i + y^i_{\lambda\mu}\partial^\mu_i) = u^\lambda d_\lambda.\end{aligned} \tag{1.35}$$

Given the splitting (1.34), the Lie derivative $L_{\overline{u}}$ along the vector field $\overline{u}$ acts on Lagrangian (1.10) by the rule

$$L_{\overline{u}}L = (L_{\overline{u}_V}\mathcal{L})\omega + [\overline{u}_H, L]_{FN}. \tag{1.36}$$

Let us assume that a Lagrangian L is invariant under the principal morphisms and world atlas transformations. We then obtain that

$$L_{\overline{u}_V}\mathcal{L} = 0 \tag{1.37}$$

and that the Lie derivative (1.36) is reduced to the horizontal differential

$$L_{\overline{u}}L = [\overline{u}_H, L]_{FN} = d_\lambda(u^\lambda\mathcal{L})\omega = d_H(\overline{u}_H \rfloor L)$$

for each principal vector field u. Moreover, one can check that

$$L_{\overline{u}}\mathcal{E}(L) = \mathcal{E}(L_{\overline{u}}L).$$

It follows that a principal vector field u is a generator of symmetry of the Euler-Lagrange operator $\mathcal{E}(L)$:

$$L_{\overline{u}}\mathcal{E}(L) = 0.$$

However, let us emphasize that expression (1.35) for a vertical part of the lift of a principal vector field is formal and its substitution into equation (1.37) makes no sense if we have no canonical form of a principal vector field. To solve this problem, we may construct the lift of a projectable vector field on the bundle E onto the infinite order jet space $J^\infty E$. In this case, a principal vector field reads

$$\begin{aligned}u &= u_V + u_H = u^\lambda(\partial_\lambda + y^i_\lambda\partial_i + \ldots) + j^\infty u_\mathfrak{g},\\ j^\infty u_\mathfrak{g} &= \sum_{k=0}^{\infty}(j^k u_\mathfrak{g})^i_{\lambda_1\ldots\lambda_k}\partial_i^{\lambda_1\ldots\lambda_k},\\ (j^k u_\mathfrak{g})^i_{\lambda_1\ldots\lambda_k} &= d_{\lambda_k}\ldots d_{\lambda_1}u^i_\mathfrak{g},\end{aligned} \tag{1.38}$$

where $u_\mathfrak{g}$ is some principal vertical vector field. Here, we take into account the canonical isomorphism

$$\beta\colon VJ^kE \to J^kVE. \tag{1.39}$$

For finite order jet calculations, we can use the formal projection of the field (1.38) on the finite order jets. In particular, the relation (1.37) then is reduced to the gauge invariance law (1.26).

1.3 Multimomentum Hamiltonian Formalism

The relation (1.29) shows that, gauge theory possesses constraints. To examine field systems with constraints, we apply the multimomentum Hamiltonian formalism [KOL, KRU] generalized to degenerate systems [ZAK].

We here assume that a world manifold X is n-dimensional. In the case of n=1, the multimomentum Hamiltonian formalism is reduced to the familiar Hamiltonian formalism.

Given a bundle E, let us consider the Legendre bundle Π (1.11) and the commutative diagram

$$\begin{tikzcd} & E \arrow[dl, "\Gamma_\Phi"'] \arrow[dr, "s_0"] & \\ J^1E & & \Pi \arrow[ll, "\Phi"] \end{tikzcd}$$

where s_0 is the global zero section of the Legendre bundle Π and Φ is a bundle morphism of Π to J^1E over E. Then,

$$\Gamma_\Phi = \Phi\circ s_0\colon\ E \to J^1E$$

is some connection on E associated with Φ. We call Φ a momentum morphism. A momentum morphism can be canonically identified with the vector-valued horizontal 1-form

$$\theta_1\circ\Phi\colon\ \Pi \to T^*X \underset{E}{\otimes} TE$$

on Π which we denote by the same symbol Φ. In the standard coordinates (1.12) on Π, we have

$$\begin{aligned}\Phi(p) &= dx^\lambda\otimes(\partial_\lambda + \Phi^i_\lambda(x^\lambda, y^i, p^\lambda_i)\partial_i),\\ \Gamma_\Phi(y) &= dx^\lambda\otimes(\partial_\lambda + \Phi^i_\lambda(x^\lambda, y^i, 0)\partial_i).\end{aligned}$$

We use p to denote elements of the Legendre bundle Π. By Π^1, we further denote the bundle $\Pi \to X$.

Given the Legendre bundle Π, there is the canonical inclusion

$$\Pi = \overset{n}{\wedge} T^*X \otimes TX \underset{E}{\otimes} V^*E \to \overset{n+1}{\wedge} T^*E \otimes TX$$

On the image of the Legendre morphism (1.55), we have

$$
\begin{aligned}
H_S|_Q = H_L &= p_m^{[\mu\lambda]} dk_\lambda^m \wedge \omega_\mu + \frac{1}{2} p_m^{[\mu\lambda]} c_{nl}^m k_\lambda^n k_\mu^l \omega \\
&\quad - \frac{\varepsilon^2}{4} a_G^{mn} g_{\mu\nu} g_{\lambda\beta} \bar{p}_m^{[\mu\lambda]} \bar{p}_n^{[\nu\beta]} \sqrt{|g|}\,\omega, \\
D_\lambda p^{[\mu\lambda]}(x) &= 0, \\
\partial_\lambda k_\mu^m + \partial_\mu k_\lambda^m &= -S_{\mu\lambda}^m(x).
\end{aligned}
$$

The last equation represents the gauge condition.

1.4 Geometry of Spontaneous Symmetry Breaking

Spontaneous symmetry breaking is the quantum phenomenon. In classical field theory, spontaneous symmetry breaking is modelled by a classical Higgs field. In geometric terms, the necessary condition of spontaneous symmetry breaking consists in reduction of a structure group of a principal bundle to its exact symmetry subgroup [IVA, TRA 1984, NIK]. We assume that a world manifold X satisfies the necessary global topological conditions for such reduction to take place.

Let H be a Lie subgroup of a structure group G of a principal bundle (P, π_P, X, G). One says that the structure group G of P is reducible to H if there exists a reduced subbundle (P^h, π_P, X, H) of P with the structure group H.

The structure group G of a principal bundle P is reducible to H if and only if there is an atlas

$$\Psi^h = \{z_\kappa^h\}$$

of P with H-valued transition functions. Given a reduced H-subbundle P^h, each atlas $\Psi^P = \{z_\kappa\}$ of P with the local sections z_κ taking their values in the total space P^h represents such an atlas Ψ^h.

Remark. Different reduced subbundles P^h and $P^{h'}$ of a principal bundle P are not isomorphic to each other in general. For example, if P is trivial, its reduced subbundle P^h fails to be trivial in general.

We further assume that H is a closed subgroup of G. Then, the structure group G of P is reducible to H if and only if there exists a global section h of the associated bundle

$$(\Sigma, \pi_{\Sigma X}, X, G, G/H)$$

with the standard fibre G/H on which the group G acts on the left. Its total space is the quotient

$$\Sigma = P/H.$$

Given a reduced subbundle P^h, the corresponding section h is defined by the relation

$$\pi_{P\Sigma} P^h = h(\pi_P P^h)$$

where $\pi_{P\Sigma}$ is the canonical projection of P onto P/H. There is 1:1 correspondence between the reduced H-subbundles P^h of the principal bundle P and the global sections h of the bundle Σ.

Remark. A closed subgroup H of a finite-dimensional Lie group is a Lie subgroup. We further assume that $\dim H > 0$.

We call Σ a Higgs bundle and its global section h a Higgs field. We use σ to denote points of Σ (and points of the quotient G/H if there is no danger of confusion).

In comparison with matter fields and gauge potentials, Higgs fields possess the following features.

(i) A group G acts transitively on the quotient space G/H, that is, for any two points σ and σ' of G/H, there exists an element $g \in G$ so that

$$\sigma' = g\sigma.$$

It follows that, given an atlas Ψ^Σ of the bundle Σ, the field functions $(\psi_\kappa^\Sigma h)$ of a Higgs field h admit the decomposition

$$(\psi_\kappa^\Sigma h)(x) = h_\kappa(x)\sigma_0, \qquad h_\kappa(x) \subset G(U_\kappa), \tag{1.57}$$

where σ_0 is the H-stable center of the quotient space G/H. The fields $h_\kappa(x)$ in this decomposition are called the Goldstone fields. In the case of internal symmetries, they however fail to be dynamic variables because one can always bring the field functions (1.57) to the H-stable form

$$(\psi_\kappa^{h\Sigma} h)(x) = \sigma_0$$

with respect to the atlas $\Psi^{h\Sigma}$ associated with Ψ^h.

(ii) Higgs fields fail to form a vector space or an affine space. At the same time, for small Goldstone deviations of σ_0 , we can write

$$\left(\psi_\kappa^{h\Sigma} h'\right)(x) = h'_\kappa(x)\sigma_0 = \exp[\varepsilon(x)]\sigma_0 \approx \sigma_0 + \varepsilon(x)\sigma_0$$

where $\varepsilon(x)$ are $\mathfrak{g}$-valued functions on U_κ. In the first order in deviations, the functions

$$\varepsilon(x) = \varepsilon^\alpha(x) K_\alpha \in \mathfrak{g} \backslash \mathfrak{h}$$

form a vector space in the sense that

$$\exp[\varepsilon(x)]\exp[\varepsilon'(x)]\sigma_0 \approx \exp[\varepsilon(x)+\varepsilon'(x)]\sigma_0.$$

By K_α, we here denote the basis elements for the Lie algebra $\mathfrak{g}$ which supplement basis elements J_m for its Lie subalgebra $\mathfrak{h}$. In some cases, small Goldstone deviations can form a vector space in the higher order in group parameters.

Let H be a Cartan subgroup of G, that is, its generators J and K obey the commutation relations

$$[J,J']\in\mathfrak{h}, \quad [K,K']\in\mathfrak{h}, \quad [K,J]\in\mathfrak{g}\backslash\mathfrak{h}.$$

There is a neighborhood of the unit element 1_G of the group G such that its elements can be expressed as

$$\exp[\varepsilon^\alpha K_\alpha]\exp[a^n J_n]. \tag{1.58}$$

In the second order in ε, we then can write

$$\begin{aligned}\exp[\varepsilon_1^\alpha K_\alpha]\exp[\varepsilon_2^\alpha K_\alpha]\sigma_0 &\approx \exp[(\varepsilon_1^\alpha+\varepsilon_2^\alpha)K_\alpha]\exp(\frac{1}{2}\varepsilon_1^\alpha\varepsilon_2^\alpha[K_\alpha,K_\beta])\sigma_0\\ &= \exp[(\varepsilon_1^\alpha+\varepsilon_2^\alpha)K_\alpha]\sigma_0.\end{aligned}$$

It follows that, in the case of a Cartan subgroup $H\subset G$, small Goldstone deviations form a vector space in the second order in group parameters. The representation of the group G on ε however is nonlinear. It is given by relations

$$\begin{aligned}K_\gamma:\ \varepsilon^\alpha K_\alpha \to \varepsilon'^\alpha K_\alpha &= K_\gamma + \sum_{i=1} c_{2i}[\underset{2i}{\cdot\cdot\cdot}[K_\gamma,\varepsilon^\alpha K_\alpha],\ldots,\varepsilon^\alpha K_\alpha]\\ &\quad -\sum_{j=1} c_j[\underset{j}{\cdot\cdot\cdot}[\varepsilon^\alpha K_\alpha, a^n J_n],\ldots,a^n J_n],\\ a^n J_n &= \sum_{i=1} c_{2i-1}[\underset{2i-1}{\cdot\cdot\cdot}[K_\gamma,\varepsilon^\alpha K_\alpha],\ldots,\varepsilon^\alpha K_\alpha],\\ J_n:\ \varepsilon^\alpha K_\alpha \to \varepsilon'^\alpha K_\alpha &= 2\sum_{i=1} c_{2i-1}[\underset{2i-1}{\cdot\cdot\cdot}[J_n,\varepsilon^\alpha K_\alpha],\ldots,\varepsilon^\alpha K_\alpha]\end{aligned} \tag{1.59}$$

where coefficients c_i are defined by the recurrence formula

$$\frac{n}{(n+1)!} = \sum_{i=1} n\frac{c_i}{(n+1-i)!}.$$

There are different types of spontaneous symmetry breaking. Here, we examine the case when matter fields possess only exact symmetries. One faces this type of spontaneous symmetry breaking in the models of so-called nonlinear realizations [COL] and in the gauge gravitation theory (Section 2.2).

Let matter fields ϕ be described by global sections of a vector bundle

$$\pi\colon E\to X$$

with the structure group H and a standard fibre F on which H acts on the left. Given a reduced H-subbundle P^h of the principal bundle P and the corresponding section h of the Higgs bundle Σ, we say that sections of E describe matter fields in the presence of a Higgs field h if E is associated with the reduced H-subbundle P^h of P. We denote such a bundle by E^h and its sections by ϕ_h.

The bundle E^h is isomorphic to the quotient $(P^h\times F)/H$ of the product $P^h\times F$ by identification of elements (p,v) and $(pg,g^{-1}v)$ for all $g\in H$. A global section ϕ_h of E^h is determined by a F-valued equivariant function f_ϕ on P^h. The bundle E^h is provided with atlases associated with atlases Ψ^h of the principal bundle P.

Remark. Let E be a bundle associated with a principal bundle P and P^h be some reduced subbundle of P. If we restrict ourselves to atlases of E associated with atlases Ψ^h of P, a bundle E can be regarded as the bundle

$$E^h = E = (P\times F)/G = (P^h\times F)/H$$

associated with the principal bundle P^h.

Given the matter fields ϕ_h, gauge potentials of these fields are represented by principal connections A on P^h. A principal connection A on the reduced subbundle P^h is uniquely extended to a principal connection A^h on P by the rule

$$(r_g)_*\tau\rfloor A^h|_{pg} = \tau\rfloor(\mathrm{ad}\,g^{-1})A|_p, \quad p\in P^h, \quad \tau\in TP. \tag{1.60}$$

Obviously, the field h is parallel with respect to A^h. With respect to an atlas Ψ^h, the local connection 1-form A^h_κ takes its values in the Lie algebra $\mathfrak{h}$.

Matter fields in the presence of different Higgs fields h and h' are described by sections ϕ_h and $\phi_{h'}$ of the matter bundles E^h and $E^{h'}$ which are associated with the different H-subbundles P^h and $P^{h'}$ of the principal bundle P. For instance, the sum $\phi_h+\phi_{h'}$ makes no sense. Moreover, a principal connection A on a reduced subbundle P^h is extended to a principal connection A^h on P which fails to induce a connection on a reduced subbundle $P^{h'\neq h}$. It follows that matter fields and gauge potentials which possess only exact symmetries must be regarded only in a pair with a certain Higgs field and the totality of matter fields and Higgs fields can not be represented by sections of the bundle product

$$\Sigma\underset{X}{\times}E$$

of the Higgs bundle and some matter bundle E. We describe the totality of ϕ-h pairs in the following way.

The total space of the principal bundle P is the total space of the principal bundle

$$\pi_{P\Sigma}\colon P \to \Sigma$$

with the structure group H. We denote this bundle by P^H.

Remark. If a principal bundle P is trivial, the bundle P^H fails to be trivial in general.

Each reduced subbundle P^h of P is the portion of P^H over $h(X) \subset \Sigma$. Given h, an atlas $\{z^H(\sigma)\}$ of the principal bundle P^H induces the atlas

$$\left\{z^{Hh}(x) = z^H(h(x))\right\}$$

of the bundles P^h and P. A principal connection A^H on P^H induces a principal connection on P^h which is extended to a certain principal connection A^{Hh} on P. Note that

$$A^{Hh} \neq A^{Hh'}$$

if $h \neq h'$.

Let

$$\pi_{E\Sigma}\colon E^H \to \Sigma$$

be a P^H-associated vector bundle with the standard fibre F. There is an isomorphism of the matter bundle E^h to the portion

$$\pi^{-1}_{E\Sigma}(h(X))$$

of E^H over $h(X)$. In particular, each global section $\phi^H(\sigma)$ of E^H defines some global section

$$\phi_h(x) = (\phi^H \circ h)(x)$$

of the bundle E^h.

The total space of the bundle E^H is the total space of the composite bundle

$$\pi_{EX} = \pi_{\Sigma X} \circ \pi_{E\Sigma}\colon E^H \to \Sigma \to X \tag{1.61}$$

which we denote by $\widetilde{E}$. Its standard fibre is the total space $\mathbf{Q}$ of the bundle which is associated with the principal bundle

$$\pi_G\colon G \to G/H \tag{1.62}$$

and possesses the standard fibre F. So, $\widetilde{E}$ is not a vector bundle and this is not associated with a principal bundle.

Remark. One can define an induced representation $H^\uparrow G$ of the group G on the space $\mathbf{Q}$, but such representation fails to be canonical and depends on choosing

representers of the cosets $\sigma \in G/H$ in G (see below). At the same time, action of the group H on $\mathbf{Q}$ is canonical. Therefore, if we restrict our consideration to atlases of $\widetilde{E}$ associated with atlases Ψ^h of the principal bundle P for some h, we can regard the composite bundle $\widetilde{E}$ as the bundle $\widetilde{E}^h$ associated with the principal bundle P^h:

$$\begin{aligned}\widetilde{E} = E^H &= (P \times F)/H = \left([(P^h \times G)/H] \times F\right)/H \\ &= \left(P^h \times [(G \times F)/H]\right)/H = (P^h \times \mathbf{Q})/H = \widetilde{E}^h.\end{aligned}$$

The space $\left([(P^h \times G)/H] \times F\right)/H$ is the quotient of the product space $P^h \times G \times F$ by the equivalence relation

$$\begin{aligned}&R_1\colon\ (p,g,v) \cong (ph_1, h_1^{-1}gh_2, h_2^{-1}v),\\ &p \in P^H \quad g \in G, \quad v \in F, \quad h_1, h_2 \in H.\end{aligned}$$

The space $\left(P^h \times [(G \times F)/H]\right)/H$ is the quotient of $P^h \times G \times F$ by the equivalence relation

$$R_2\colon\ (p,g,v) \cong (ph_4, h_4^{-1}gh_3h_4, h_4^{-1}h_3^{-1}v).$$

Given $h_3, h_4 \in H$, one can choose

$$h_1 = h_4, \qquad h_2 = h_3h_4,$$

and vise versa. It follows that the R_1-equivalent elements of the product $P^h \times G \times F$ are also R_2-equivalent, and vise versa.

Remark. Let the bundle (1.62) be trivial. The reduced subbundles P^h of the principal bundle P then are isomorphic to each other and, for any h, the principal bundle P^H is isomorphic to the pull-back $(\pi_{\Sigma X})^* P^h$ of the bundle P^h by the projection $\pi_{\Sigma X}$. Given an atlas

$$\Psi^h = \left\{U_\kappa, \psi^h_\kappa\right\}$$

of P, the bundle P^H can be provided with the pull-back atlas

$$\Psi^{hH} = \left\{U^H_\kappa = \pi^{-1}_{\Sigma X}(U_\kappa), \psi^H_\kappa = \psi^h_\kappa \circ \mathring{\pi}_{\Sigma X}\right\}$$

where $\mathring{\pi}_{\Sigma X}$ is the bundle morphism (0.1) associated with $\pi_{\Sigma X}$. Every principal connection A on P^h induces a certain pull-back principal connection on P^H given by the principal connection form $(\mathring{\pi}_{\Sigma X})^*\overline{A}$. With respect to the pull-back atlases of the bundle P^H, this form is constant on fibres of the Higgs bundle Σ and is independent of vertical tangent vectors to Σ. At the same time, the principal-associated structures $\widetilde{E}^h$ and $\widetilde{E}^{h'}$ of the bundle $\widetilde{E}$ fail to be equivalent. Given the atlases Ψ^h and $\Psi^{h'}$ of the principal bundle P, there exists no atlas of the bundle $\widetilde{E}$ which would be associated with the union atlas $\Psi^h \cup \Psi^{h'}$ of P possessing G-valued transition functions. Thus, the bundle $\widetilde{E}$ has no canonical structure of a H-bundle. For instance, the bundle

$$\widetilde{P} = P \to \Sigma \to X \tag{1.63}$$

is not isomorphic to the principal bundle P. In particular, the canonical action (0.4) of G on P fails to keep the fibration (1.63).

For a global section h of the bundle Σ and a global section ϕ^H of the bundle E^H, their composite

$$\widetilde{\phi} = \phi^H \circ h \tag{1.64}$$

is a global section of the bundle $\widetilde{E}$. Conversely, each section $\widetilde{\phi}$ of the bundle $\widetilde{E}$ is represented by some composite (1.64). In particular, the sections (1.64) with the same projection

$$h = \pi_{E\Sigma} \circ \widetilde{\phi}$$

exhaust all sections of the bundle E^h. We thus may describe the totality of ϕ-h pairs by global sections of the composite bundle $\widetilde{E}$. Their configuration space then is the jet manifold $J^1\widetilde{E}$ of the bundle $\widetilde{E}$, and their momentum space is the Legendre manifold

$$\Pi = \overset{n}{\wedge} T^*X \otimes TX \otimes V^*\widetilde{E}. \tag{1.65}$$

Since the bundle $\widetilde{E}$ fails to be associated with a principal bundle, it does not admit an associated principal connection. Given principal connections on the bundles E^H and Σ, we however can construct a general connection

$$\Gamma\colon \widetilde{E} \to J^1\widetilde{E}$$

on $\widetilde{E}$ by means of the canonical morphism

$$\xi\colon J^1\Sigma \underset{\Sigma}{\times} J^1E^H \to J^1\widetilde{E}. \tag{1.66}$$

This morphism is defined by the relation

$$\xi(j^1_x h, j^1_{h(x)}\phi^H) = j^1_x(\phi^H \circ h)$$

for every section (1.64) of the bundle $\widetilde{E}$. Let us write its coordinate expression.

Let $\Psi^P = \{z_\kappa\}$ be an atlas of the principal bundle P and $\Psi^H = \{z^H_k\}$ be an atlas of the principal bundle P^H. We provide the total spaces Σ and $E^H = \widetilde{E}$ with the bundle coordinates

$$\Sigma \ni \sigma = (x^\lambda, \sigma^m), \qquad \sigma^m(\sigma) = (a^m \circ \psi^P_\kappa \circ z^H_k)(\sigma),$$
$$\widetilde{E} = E^H \ni y = (x^\lambda, \sigma^m, y^i), \qquad y^i = (v^i \circ \psi^H_k)(y),$$

where (x^μ) are coordinates on the base X, (a^m) are parameters of the group G, and (v^i) are coordinates on the standard fibre F.

Remark. The coordinate $\sigma^m(\sigma)$ of an element $\sigma \in \Sigma$ are group parameters of the representer $(\psi_\kappa \circ z^H_k)(\sigma)$ of the coset $\psi^\Sigma_\kappa(\sigma)$ in the group G. The coordinates σ^m on the Higgs bundle Σ are bundle coordinates, but they are not associated with any bundle atlas of Σ. We have

$$\Sigma_x \to P_x \to G$$

by the relation (1.60) instead of

$$\Sigma_x \to G/H \to G$$

by the relation (1.57).

Let us provide the jet manifolds $J^1\Sigma$, J^1E^H, and J^1E with the adapted coordinates

$$J^1\Sigma \ni s = (x^\lambda, \sigma^m, \sigma^m_\lambda),$$
$$J^1E^H \ni w = (x^\lambda, \sigma^m, y^i, y^i_\lambda, y^i_m),$$
$$J^1\widetilde{E} \ni q = (x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, \widetilde{y}^i_\lambda).$$

In this coordinates, the morphism (1.66) is given by the expression

$$(x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, \widetilde{y}^i_\lambda) \circ \xi = (x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, y^i_m\sigma^m_\lambda + y^i_\lambda).$$

To get this expression, we can use the canonical contact map θ_1 of the jet manifolds $J^1\Sigma$, J^1E^H and $J^1\widetilde{E}$ onto affine subbundles of the bundles $T^*X \otimes_X T\Sigma$, $T^*\Sigma \otimes_\Sigma TE$ and $T^*X \otimes_X T\widetilde{E}$ respectively. Then, the morphism (1.66) results from the diagram

$$\begin{array}{ccc} J^1\Sigma \times J^1E^H & \xrightarrow{\xi} & J^1\widetilde{E} \\ {\scriptstyle\theta_1}\big\downarrow & & \big\downarrow{\scriptstyle\theta_1} \\ (T^*X \otimes T\Sigma) \times (T^*\Sigma \otimes TE^H) & \xrightarrow{\rfloor} & T^*X \otimes T\widetilde{E} \end{array} \tag{1.67}$$

In adapted coordinates, this diagram reads

$$[dx^\lambda \otimes (\partial_\lambda + \sigma^m_\lambda\partial_m)] \rfloor [dx^\lambda \otimes (\partial_\lambda + y^i_\lambda\partial_i) + d\sigma^m \otimes (\partial_m + y^i_m\partial_i)]$$
$$= dx^\lambda \otimes [\partial_\lambda + \sigma^m_\lambda\partial_m + (y^i_\lambda + \sigma^m_\lambda y^i_m)\partial_i].$$

Now, let Γ and A^H be associated principal connections on the bundles Σ and E^H respectively. They have the coordinate expressions

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^m_\lambda(\sigma)\partial_m),$$
$$A^H = dx^\lambda \otimes (\partial_\lambda + A^i_\lambda(y)\partial_i) + d\sigma^m \otimes (\partial_m + A^i_m(y)\partial_i).$$

From the diagram (1.67), we then obtain a connection $\widetilde{\Gamma}$ on the bundle $\widetilde{E}$:

$$\widetilde{\Gamma}\widetilde{E} = \xi\left((\Gamma\circ\pi_{E\Sigma})E^H \times A^H E^H\right),$$
$$\widetilde{\Gamma} = dx^\lambda \otimes [\partial_\lambda + \Gamma^m_\lambda(\pi_{E\Sigma}(y))\partial_m + (A^i_m(y)\Gamma^m_\lambda(\pi_{E\Sigma}(y)) + A^i_\lambda(y))\partial_i].$$

This connection fails to be an associated principal connection because of the term

$$A^i_m(y)\Gamma^m_\lambda(\pi_{E\Sigma}(y))\,dx^\lambda \otimes \partial_i. \tag{1.68}$$

This term disappears if the connection A^H on E^H is a pull-back connection. If the bundle (1.62) is no trivial, the bundle E^H however fails to admit a pull-back connection.

Given a connection $\widetilde{\Gamma}$ on $\widetilde{E}$, one can define a multimomentum Hamiltonian form H on the Legendre manifold Π (1.65).

Let the Legendre bundle $\Pi \to \widetilde{E}$ be provided with the local standard coordinates $(x^\lambda, q^a, p^\lambda_a)$ where we introduce the condensed notation

$$q^a = (\sigma^m, y^i), \qquad p^\lambda_a = (p^\lambda_m, p^\lambda_i).$$

Remark. To define the local coordinates p^λ_m, one can use the local morphisms

$$\alpha\circ Vz^H_\kappa\colon V\Sigma \to VP \to P\times \mathfrak{g}$$

where α denotes the canonical trivial vertical splitting (0.16) of the principal bundle P and Vz^H_κ are the vertical tangent morphisms to the local sections z^H_κ associated with an atlas Ψ^H of the principal bundle P^H.

The multimomentum Hamiltonian form (1.44) associated with a connection $\widetilde{\Gamma}$ on the bundle $\widetilde{E}$ reads

$$H_\Gamma = \widetilde{\Gamma}\rfloor\theta\colon \Pi \to \overset{n}{\wedge} T^*\widetilde{E},$$
$$H_\Gamma = p^\lambda_a dq^a \wedge \omega_\lambda - p^\lambda_a\widetilde{\Gamma}^a_\lambda\omega.$$

Other multimomentum Hamiltonian forms on Π then can be written as

$$H = H_\Gamma - \mathcal{H}\omega \tag{1.69}$$

where $\mathcal{H}\omega$ is some exterior horizontal n-form on the bundle Π^1.

Given a multimomentum Hamiltonian form H, the Hamiltonian equations for a local section $(q^a(x), p^\lambda_a(x))$ of the bundle Π^1 read

$$\partial_\mu p^\mu_b = -p^\lambda_a\partial_b\widetilde{\Gamma}^a_\lambda - \partial_b\mathcal{H},$$
$$\partial_\mu q^b = \widetilde{\Gamma}^b_\mu + \partial^b_\mu\mathcal{H}.$$

For a matter field, the Hamiltonian equations take the form

$$\partial p^\mu_i = -p^\lambda_j\partial_i\widetilde{\Gamma}^j_\lambda - \partial_i\mathcal{H},$$
$$\partial_\mu y^i = \widetilde{\Gamma}^i_\mu + \partial^i_\mu\mathcal{H}.$$

These equations differ from Hamiltonian equations for matter fields in the presence of a background Higgs field h because of the term (1.68) in the connection $\widetilde{\Gamma}^i_\mu$. This term is treated to describe the additional contribution of Higgs fields h into the covariant differential of matter fields. Let us note that, to construct multimomentum Hamiltonian form (1.69), one must require the Hamiltonian density $\mathcal{H}$ be invariant under gauge maps induced by gauge morphisms of the bundle $\widetilde{E}$. Since the bundle $\widetilde{E}$ is not associated with a principal bundle, gauge morphisms of $\widetilde{E}$ fail to be principal morphisms in general. For instance, we have two types of atlas transformations. They are induced by transformations of atlases Ψ^P of the principal bundle P and transformations of atlases Ψ^H of the principal bundle P^H.

Gauge bundle morphisms of the bundle $\widetilde{E}$ are associated general principal morphisms of the bundle E^H which keep the fibration (1.61) of the bundle $\widetilde{E}$. They are generated by general principal isomorphisms Φ_H of the principal bundle P^H which are projected to principal morphisms of the bundle Higgs Σ. For example, principal isomorphisms (0.6) of the principal bundle P are also general principal isomorphisms

$$\Phi_P\colon P^H_\sigma \to P^H_{\sigma'},$$
$$\sigma = \pi_{P\Sigma}(p) = \pi_{P\Sigma}(pg_H), \qquad g_H \in H,$$
$$\sigma' = \pi_{P\Sigma}(pf_s(p)) = \pi_{P\Sigma}(pg_H f_s(pg_H)) \tag{1.70}$$

of the principal bundle P^H which are projected to associated principal morphisms of the Higgs bundle Σ and to the identity morphism of the base X. The isomorphisms (1.70), in turn, yield associated general principal morphisms of the bundle E^H which are also gauge bundle morphisms of the composite bundle $\widetilde{E}$.

For instance, let the bundle P^H be trivial and $z^H(\sigma)$ be its global section. Given a global section h of the Higgs bundle Σ, we have a global section

$$z(x) = z^H(h(x))$$

of the bundle P. Moreover, for every $\sigma\in\Sigma$, there exists an element $g_\sigma \in G$ such that

$$z^H(\sigma) = z(\pi_{\Sigma X}(\sigma))g_\sigma.$$

Given a global section $z(x)$ of the principal bundle P, one can define the embedding

$$G \ni g \to \Phi_g \in G(X)$$

where

$$\Phi_g\colon P_x = z(x)G \to z(x)gG, \qquad g\in G.$$

Given a global section $z^H(\sigma)$ of Σ, we can write

$$\Phi\colon\ P^H_\sigma = z^H(\sigma)H = z(x)g_\sigma H \to z(x)gg_\sigma H$$
$$= z(x)g_{\sigma'}(g_{\sigma'}^{-1}gg_\sigma)H = z^H(\sigma')g_H H, \quad x = \pi_{\Sigma X}(\sigma),$$

where

$$\sigma' = \pi_{\Sigma X}(gg_\sigma), \qquad g_H = g_{\sigma'}^{-1}gg_\sigma \in H.$$

This is expression of the induced representation $H^\uparrow G$ of the group G on elements of P^H when representers of the cosets $\sigma \in \Sigma$ are chosen to be elements $z^H(\sigma)$. This representation can be transferred to field functions of sections of the associated bundle E^H:

$$g : \phi^H(\sigma) = f_\phi(z^H(\sigma)) \to f_\phi(z^H(\sigma')g_H) = g_H^{-1}\phi^H(\sigma')$$

and to field functions of sections of the bundle $\widetilde{E}$:

$$\widetilde{\phi} = \phi^H(\sigma(x)) \to \widetilde{\phi}'(x) = g_H^{-1}\phi^H(\sigma'(x)).$$

Remark. Let us be given a group G, its subgroup H and a space F on which the subgroup H acts on the left. Let $f(g)$ be F-valued functions on G satisfying the following condition

$$f(gg_H) = g_H^{-1}f(g), \qquad g_H \in H. \tag{1.71}$$

The induced representation $H^\uparrow G$ of the group G [MAC] is defined to be the action of G on $f(g)$ by the rule

$$G \ni g'\colon\ f(g) \to f(g'g).$$

In accordance with the relation (1.71), the induced representation can be defined on F-valued functions $\widetilde{f}(\sigma)$ on the quotient space G/H. Given a global section $z(\sigma)$ of the bundle (1.62), such representation reads

$$G \ni g\colon\ \widetilde{f}(\sigma) = f(z(\sigma)) \to f(gz(\sigma)) = f(z(\sigma')g_H)$$
$$= g_H^{-1}f(z(\sigma')) = g_H^{-1}\widetilde{f}(\sigma'), \qquad g \in H,$$

where

$$\sigma' = \pi_G(gz(\sigma)).$$

Obviously, this representation depends on option of representers $z(\sigma) \in G$ of the cosets $\sigma \in G/H$.

Remark. The above-mentioned models of nonlinear realizations [COL] exemplify the induced representations when H is a Cartan subgroup of G and $\widetilde{f}(\sigma)$ are constant functions

$$\widetilde{f}(\sigma) = (\sigma, v).$$

For the coset class σ of each element (1.58), one chooses the representer

$$\exp[\varepsilon^\alpha K_\alpha]$$

and provides σ with the coordinates $\{\varepsilon^\alpha\}$. The action of the group G on (ε^α, v) then is defined to be

$$G \ni g\colon\ (\varepsilon^\alpha, v) \to (\varepsilon'^\alpha, \exp[a^n J_n]v)$$

where

$$g\ \exp[\varepsilon^\alpha K_\alpha] = \exp[\varepsilon'^\alpha K_\alpha]\exp[a^n J_n]$$

and parameters ε'^α and a^n are given by relations (1.59). In the second order in ε, we have

$$J_n\colon\ \varepsilon^\alpha \to c^\alpha_{n\nu}\varepsilon^\nu, \qquad a^m = \delta^m_n,$$
$$K_\gamma\colon\ \varepsilon^\alpha \to \delta^\alpha_\gamma + \frac{1}{12}(c^\beta_{\gamma\nu}c^\alpha_{\beta\kappa} - 3c^n_{\gamma\nu}c^\alpha_{\kappa n})\varepsilon^\nu\varepsilon^\kappa, \qquad a^m = \frac{1}{2}c^m_{\gamma\nu}\varepsilon^\gamma,$$

where c are the structure constants of the group G.

Let us note that Lagrangian and Hamiltonian densities are usually invariant under the gauge subgroup $G(\Sigma)$ of general principal morphisms (1.70), but fail to be invariant under all these morphisms in general. In this case, Goldstone components of a Higgs field can not be completely removed by gauge transformations.

Chapter 2

GAUGE THEORY OF CLASSICAL GRAVITY

The gauge gravitation theory is based on the geometric equivalence principle formulated in Introduction. There are two physical underlying reasons of this equivalence principle. These are the Dirac fermion matter and the space-time structure. One attempts to unify them in the framework of the twistor theory [PEN]. We here do not discuss this problem. We start from Dirac fermion fields. A key point consists in the fact that, to construct the Dirac operator, one must define representation of cotangent vectors to a world manifold by Dirac's γ-matrices [SAR 1991].

We further assume that a 4-dimensional world manifold X satisfies certain topological conditions formulated in Section 2.3.

2.1 Dirac Fermion Fields

We describe Dirac fermion fields as follows [BUG, CHI]. Given a Minkowski space M with the Minkowski metric η, let

$$A_M = \underset{n}{\otimes} M^n, \qquad M^0 = \mathbb{R}, \qquad M^{n>0} = \overset{n}{\otimes} M,$$

be the tensor algebra modelled on M. The complexified quotient of this algebra by the two-sided ideal generated by elements

$$e \otimes e' + e' \otimes e - 2\eta(e, e') \in A_M, \qquad e \in M,$$

forms the complex Clifford algebra $\mathbb{C}_{1,3}$. A spinor space V is defined to be a linear space of some minimal left ideal of $\mathbb{C}_{1,3}$ on which this algebra acts on the left. We then have the representation

$$\gamma\colon\ M \otimes V \to V \tag{2.1}$$

of elements of the Minkowski space $M \subset \mathbb{C}_{1,3}$ by γ-matrices on V:

$$\widehat{e}^a v^A = \gamma(e^a \otimes v^A) = \gamma^{aA}{}_B v^B$$

where

$$\{e^a, a = 0, 1, 2, 3\}, \quad \eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1,$$

is a fixed basis for M, $\{v^A\}$ is a basis for V, and γ^a are Dirac's matrices of a fixed form.

Let us consider the transformations preserving the representation (2.1). These are pairs (l, l_s) of Lorentz transformations l of the Minkowski space M and invertible elements l_s of $\mathbb{C}_{1,3}$ such that

$$lM = l_s M l_s^{-1},$$
$$\gamma(lM \otimes l_s V) = l_s \gamma(M \otimes V).$$

Elements l_s form the Clifford group whose action on M however is not effective. We here restrict ourselves to its spinor subgroup $L_s = SL(2, \mathbb{C})$ such that

$$L = SO(3,1) = L_s/\mathbb{Z}_2.$$

Remark. By $SO(3,1)$, we denote the connected Lorentz group. On M, this group is represented by matrices l with

$$\det l = 1, \quad l^0{}_0 > 0.$$

Remark. Generators of the spinor group L_s act on V by the representation

$$I_{ab} = \frac{1}{4}[\gamma_a, \gamma_b].$$

Since

$$(I_{ab})^+ \neq -I_{ab},$$

the standard Hermitian metric in V fails to be L_s-invariant. At the same time, we have

$$(\gamma^0)^+ = \gamma^0, \quad (\gamma^0\gamma^a)^+ = \gamma^0\gamma^a, \quad (I_{ab})^+\gamma_0 = -\gamma_0 I_{ab}.$$

Hence, the L_s-invariant spinor metric in V can be defined as [CRO]

$$\mathrm{a}(v, v) = v^+\gamma^0 v. \tag{2.2}$$

Let E_M be a bundle with the structure group L, the standard fibre M, and the base X so that its associated principal bundle P_M has lift to a principal bundle (P_s, X, L_s) with the structure group L_s:

$$r\colon\ P_s \to P_M = P_s/\mathbb{Z}_2,$$
$$E_M = (P_M \times M)/L = (P_s \times M)/L_s.$$

We call E_M a Minkowski space bundle and P_s a principal spinor bundle.

Remark. There exist the topological obstructions to lifting of P_M to P_s. We here do not discuss them. Later, we shall require E_M be isomorphic to the cotangent bundle T^*X, and a world manifold X will be assumed to satisfy the necessary topological conditions (Section 2.3).

Let (E, π, X, V, L_s) be a matter bundle associated with the principal bundle P_s. We call it a spinor bundle. One can define the morphism

$$\gamma_E\colon\ E_M \otimes E = (P_s \times (M \otimes V))/L_s \to (P_s \times \gamma(M \otimes V))/L_s = E.$$

Given an atlas $\{z^s_\kappa\}$ of P_s and the associated atlas

$$\{z^M_\kappa = r \circ z^s_\kappa\}$$

of P_M, this morphism reads

$$\hat{e}^a(x)v^A(x) = \gamma_E(e^a(x) \otimes v^A(x)) = \gamma^{aA}{}_B v^B(x), \qquad x \in U_\kappa,$$

where

$$\{e^a(x) = [z^M_\kappa(x)]_M e^a\}$$

and

$$\{v^a(x) = [z^s_\kappa(x)]_V v^A\}$$

are the associated bases for fibres M_x and V_x of the bundles E_M and E respectively.

Dirac fermion fields are described by global sections ϕ of the spinor bundle E provided with the representation morphism γ_E which yields the representation of sections of E_M by γ-matrices on ϕ:

$$\gamma_E\colon\ \tau(x) = \tau_a(x)e^a(x) \to \hat{\tau}(x) = \tau_a(x)\hat{e}^a(x) = \tau_a(x)\gamma^a.$$

To construct the Dirac operator on ϕ, one must require E_M be isomorphic to the cotangent bundle T^*X over a world manifold X. It takes place only if the principal bundle of linear frames

$$LX = (P, \pi_{PX}, X, GL_4)$$

contains a reduced subbundle

$$L^hX = (P^h, \pi_{PX}, X, L)$$

with the structure group L, that is, the structure group GL_4 of the bundle LX is reducible to the Lorentz group.

The geometric equivalence principle thereby is the necessary condition in order that Dirac fermion fields can be defined.

2.2 Tetrad Gravitational Fields

The structure group of the principal bundle LX is reducible to the Lorentz group if and only if there exists a global sections h of the associated bundle

$$(\Sigma, \pi_{\Sigma X}, X, GL_4/L, GL_4)$$

with the standard fibre GL_4/L. We call h a tetrad field. There is 1:1 correspondence between the tetrad fields h and the reduced L-subbundles L^hX of LX possessing the structure Lorentz group. This correspondence is given by the relation

$$\pi_{P\Sigma}P^h = h(\pi_{PX}P^h)$$

where $\pi_{P\Sigma}$ is the canonical projection of P onto

$$\Sigma = P/L.$$

The Higgs bundle Σ is the 2-fold covering of the bundle Λ of pseudo-Riemannian bilinear forms in cotangent spaces T^*_xX to X. A global section of Λ is a pseudo-Riemannian world metric g on X. A global section h of the bundle Σ defines uniquely a pseudo-Riemannian metric g.

Remark. The group space of GL_4 is homeomorphic to the topological space $\mathbb{R}P^3 \times S^3 \times \mathbb{R}^{10}$. The group space of L is $\mathbb{R}P^3 \times \mathbb{R}^3$. Here, S^3 denotes the 3-dimensional sphere, and

$$\mathbb{R}P^3 = S^3/\mathbb{Z}_2$$

is the 3-dimensional real projective space. The quotient space GL_4/L is homeomorphic to $S^3 \times \mathbb{R}^7$, and the bundle

$$GL_4 \to GL_4/L \tag{2.3}$$

is trivial. Pseudo-Riemannian metrics in $\mathbb{R}^4$ form the topological space $\mathbb{R}P^3 \times \mathbb{R}^7$.

Given a tetrad field h, let local sections $\{z^h_\kappa\}$ associated with an atlas of LX take their values in the corresponding reduced subbundle P^h. They then define an atlas Ψ^h of LX such that its transition functions are L-valued. Moreover, with respect to Ψ^h, metric functions of g come to the Minkowski metric

$$g_\kappa = \psi^h_\kappa g = \eta$$

and field functions $(\psi^h_\kappa h)(x)$ of the tetrad field h take their values in the center σ_0 of the quotient space GL_4/L.

Remark. For the sake of simplicity, we further denote an atlas Ψ of a principal bundle and associated atlases by the same symbol Ψ. By Ψ^T, we denote holonomic atlasses.

If one considers the cotangent bundle T^*X provided only with atlases Ψ^h, this bundle acquires the structure of the L-bundle M^hX of Minkowski spaces which is associated with L^hX, that is,

$$T^*X = (P \times T^*)/GL_4 = (P^h \times M)/L = M^hX$$

where T^* denotes the standard fibre of T^*X. For different h and h', the L-bundles M^hX and $M^{h'}X$ are not isomorphic. Their fibres M_x and M'_x are cotangent spaces T^*_xX, but provided with different Minkowski space structures.

The feature of a gravitational field thus is clarified. A tetrad gravitational field h itself, unlike other fields, defines reference frames Ψ^h such that, given different gravitational fields h and h', the corresponding reference frames Ψ^h and $\Psi^{h'}$ fail to be equivalent in a sense.

Since

$$h(x) = \pi_{P\Sigma}(z^h_\kappa(x)),$$

given an atlas Ψ^h and a holonomic atlas Ψ^T, a tetrad field h can be represented by the family of tetrad functions $\{h_\kappa\}$:

$$h_\kappa(x) = \psi^T_\kappa(x) z^h_\kappa(x) = [z_\kappa(x)]^{-1}_T \circ [z^h_\kappa(x)]_T, \qquad x \in U_\kappa,$$
$$(\psi^T_\kappa h)(x) = h_\kappa(x)\sigma_0. \tag{2.4}$$

The tetrad functions (2.4) define gauge transformations of the atlas

$$\{\psi^h_\kappa(x) = [z^h_\kappa(x)]^{-1}_T\}$$

to the holonomic atlas

$$\{\psi^T_\kappa(x) = [z_\kappa(x)]^{-1}_T = h_\kappa(x) \circ \psi^h_\kappa(x)\}.$$

In the index form, the tetrad functions (2.4) describe the corresponding transformations

$$t^h_a(x) = (\psi^h_\kappa(x))^{-1} t_a = (\psi^T_\kappa(x))^{-1} h^\mu_a(x) t_\mu$$
$$= h^\mu_a(x)(\psi^T_\kappa(x))^{-1} t_\mu = h^\mu_a(x)\partial_\mu, \qquad x \in U_\kappa, \tag{2.5}$$

between the bases $\{\partial_\mu\}$ and $\{t^h_a(x)\}$ associated with Ψ^T and Ψ^h for tangent spaces and between the corresponding bases

$$dx^\mu = h^\mu_a(x) t^{ha}(x)$$

for cotangent spaces.

For instance, we have

$$g_\kappa(x) = h_\kappa(x)\eta, \qquad x \in U_\kappa,$$
$$g^{\mu\nu}(x) = h^\mu_a(x) h^\nu_b(x) \eta^{ab}.$$

This splitting of metric field functions looks like the decomposition (1.57) of a Higgs field where the Minkowski metric η and tetrad functions h_κ play the role of a L-stable Higgs field and Goldstone fields respectively [IVA 1983]. However, in contrast with the internal symmetry case, Goldstone components of a gravitational field are not removed by atlas transformations because a reference frame Ψ^h fails to be holonomic in general and the associated basis elements

$$\partial_a = h^\mu_a(x)\partial_\mu$$

contain tetrad functions.

Remark. In expression (2.5), tetrad functions are represented by matrices of elements of the group GL_4 whose action on the standard fibre T of the tangent bundle is described with respect to a fixed frame $\{t_a\}$:

$$GL_4(U_\kappa) \ni h_\kappa(x): \; \{t_a\} \to \{h^\mu_a(x) t_\mu\}.$$

Given an atlas Ψ^h, one can choose representers $h'_\kappa(x)$ of the cosets $(\psi^{h\Sigma}_\kappa h)(x)$ which differ from the tetrad functions (2.4) in gauge Lorentz transformations

$$h'_\kappa(x) = h_\kappa(x) l(x).$$

This option involves additional quantities $l(x)$ in the Dirac operator.

We call h a tetrad gravitational field and g a metric gravitational field. However, only small geometric deviations of a metric field g acquire the ordinary sense of a physical field. In the first order in deviations ε, we have

$$g'^{\mu\nu} = g^{\mu\nu} + \varepsilon^{\mu\nu},$$
$$g'_{\mu\nu} = g_{\mu\nu} - \varepsilon_{\mu\nu} \approx g_{\mu\nu} - g_{\mu\alpha} g_{\nu\beta} \varepsilon^{\alpha\beta}. \tag{2.6}$$

The metric g in expression (2.6) plays the role of a background metric and, given a nonholonomic atlas Ψ^h associated with g, the notions of spin and energy-momentum of the deviations

$$\varepsilon_{ab} = h^\mu_a h^\nu_b \varepsilon_{\mu\nu}$$

are precisely defined. In general, the deviations (2.6) however fail to be superposable even in the first order in ε (Section 3.2).

We say that a spinor bundle E^h describes Dirac fermion fields ϕ_h in the presence of a gravitational field h if this bundle is associated with the L_s-lift

$$(P_s^h, \pi_{PX} r, X, L_s),$$
$$r\colon P_s^h \to P^h = P_s^h/\mathbb{Z}_2,$$

of the reduced L-subbundle L^hX of the principal bundle LX. In this case, the corresponding Minkowski space bundle E_M^h is isomorphic to the cotangent bundle T^*X regarded as the L-bundle

$$M^hX = (P_s^h \times M)/L_s.$$

We then can define the representation

$$\gamma_h\colon T^*X \otimes E^h = (P_s^h \times (M \otimes V))/L_s \to (P_s^h \times \gamma(M \otimes V))/L_s = E^h \qquad (2.7)$$

of cotangent vectors to X by Dirac's γ-matrices on elements of E^h. With respect to an atlas $\{z_\kappa^s\}$ of P_s^h and the associated atlas $\{z_\kappa^h = rz_\kappa^s\}$ of LX, the morphism γ_h reads

$$\hat{t}^{ha}(x)v^A(x) = \gamma_h(t^{ha}(x) \otimes v^A(x)) = \gamma^{aA}{}_B v^B(x)$$

where

$$\{t^{ha}(x) = [z_\kappa^h(x)]_{T^*} t^a\}$$

and

$$v^A(x) = [z_\kappa^s(x)]_V v^A$$

are the associated bases for fibres T_x^*X and V_x.

For any holonomic atlas of T^*X, we can introduce the local 1-forms

$$t^{ha}(x) = h_\mu^a(x)dx^\mu = h^a \qquad (2.8)$$

which represent components of the canonical basic soldering form (0.21):

$$\theta_X = t^{ha}(x) \otimes h_a.$$

The representation (2.7) then takes the form

$$\gamma_h(h^a) = \hat{h}^a = \gamma^a.$$

Remark. By h_μ^a, we denote components of the matrix which is inverse of h_a^μ, that is,

$$h_\mu^a h_a^\nu = \delta_\mu^\nu, \qquad h_\mu^a h_b^\mu = \delta_b^a.$$

Given a gravitational field h, each principal connection A_s on the principal bundle P_s^h associated with the spinor bundle E^h induces a principal connection A on the reduced subbundle L^hX. By the law (1.60), this connection A is uniquely extended to a principal connection Γ^h on LX so that fields h and g are parallel with respect to Γ^h. If A_s' is another principal connection on P_s^h, the corresponding connection Γ'^h differs from Γ^h in a torsion. As a consequence, principal connections A on the bundle E^h form the affine space of Lorentz gauge potentials modelled on the linear space of torsion fields.

We further call principal connections Γ on the principal bundle LX the world connections and principal connections A_s on principal spinor bundles the principal spinor connection. If $\Gamma = \Gamma^h$ for some tetrad field h, we call Γ^h the Lorentz connection.

Remark. Given a pseudo-Riemannian metric g, every world connection Γ can be decomposed in the sum of three quantities

$$\Gamma = \{\ \} + S + C$$

where $\{\ \}$ denotes the Christoffel symbols of the metric g, S is a contortion and C is called a nonmetricity. These quantities are calculated from the relations

$$\nabla g = D_{\{\ \}} g = 0,$$
$$\nabla \theta_X = D_{\{\ \}} \theta_X = 0,$$
$$Dg = -2C,$$
$$D\theta_X = \Omega$$

where $\nabla = D_{\{\ \}}$ denotes the covariant differential associated with the Christoffel connection $\{\ \}$ and Ω is the 2-form (0.51) of torsion of the connection Γ. Coefficients of forms $\{\ \}$, S and C read

$$\partial_\mu g_{\alpha\nu} - \Gamma_{\alpha\nu\mu} - \Gamma_{\nu\alpha\mu} = -2C_{\alpha\nu\mu},$$
$$\Gamma_{\alpha\nu\mu} = \{_{\alpha\nu\mu}\} + S_{\alpha\nu\mu} + C_{\alpha\nu\mu},$$
$$\{_{\alpha\nu\mu}\} = \{_{\alpha\mu\nu}\} = \frac{1}{2}(g_{\alpha\nu,\mu} + g_{\alpha\mu,\nu} - g_{\nu\mu,\alpha}),$$
$$S_{\alpha\nu\mu} = -S_{\nu\alpha\mu} = (\Omega_{\alpha\nu\mu} + \Omega_{\mu\nu\alpha} + \Omega_{\nu\mu\alpha} + C_{\mu\alpha\nu} - C_{\mu\nu\alpha}),$$
$$\Omega_{\alpha\nu\mu} = -\Omega_{\alpha\mu\nu} = \frac{1}{2}(\Gamma_{\alpha\nu\mu} - \Gamma_{\alpha\mu\nu}),$$
$$C_{\alpha\nu\mu} = C_{\nu\alpha\mu} = \frac{1}{2}(\Gamma_{\alpha\nu\mu} + \Gamma_{\nu\alpha\mu} - g_{\alpha\nu,\mu}).$$

If Γ is a metric connection (that is, g is parallel with respect to Γ), the nonmetricity term C vanishes.

Remark. Given a world connection Γ and a tetrad field h represented by tetrad functions h_μ^a, we have the following relation

$$\partial_\lambda h_a^\mu + \Gamma^\mu{}_{\nu\lambda} - \Gamma^b{}_{a\lambda} h_b^\mu = 0.$$

A tetrad field h is parallel with respect to a world connection Γ if

$$\Gamma_{ba\lambda} + \Gamma_{ab\lambda} = 0.$$

Given an associated principal spinor connection A on the bundle E^h, one can construct the Dirac operator

$$L_D = \gamma_h \circ D\colon\ J^1E^h \to T^*X \underset{E^h}{\otimes} VE^h \to VE^h \tag{2.9}$$

where D is the covariant differential (0.44). To define the operator (2.9), we use the fact that the vertical tangent bundle VE^h over the spinor bundle E^h admits the canonical vertical splitting (0.15). The bundle morphism γ_h of VE^h in expression (2.9) is the pull-back of the bundle morphism γ_h of E^h in expression (2.7).

With respect to an atlas $\{z^s_\kappa\}$ of the principal spinor bundle P^h_s and a holonomic atlas of LX, the Dirac operator (2.9) reads

$$L_D\phi_h = \widehat{dx}^\mu D_\mu\phi_h = h^\mu_a(x)\widehat{h}^a D_\mu\phi_h = h^\mu_a(x)\gamma^a D_\mu\phi$$

where h^a are forms (2.8).

Fermion fields ϕ_h and $\phi_{h'}$ in the presence of different gravitational fields h and h' are described by the spinor bundles E^h and $E^{h'}$ associated with the L_s-lifts P^h_s and $P^{h'}_s$ of the different L-subbundles L^hX and $L^{h'}X$ of the linear frame bundle LX. Since the bundle (2.3) is trivial, all reduced L-subbundles L^hX of the principal bundle LX are isomorphic to each other. It follows that the spinor bundles E^h and $E^{h'\neq h}$ also are isomorphic. The Minkowski space structures M^hX and $M^{h'}X$ of the cotangent bundle T^*X however fail to be equivalent in the sense that transition functions between the atlases Ψ^h and $\Psi^{h'}$ of T^*X are GL_4-valued. Let us compare the representations γ_h and $\gamma_{h'}$.

For two arbitrary elements q and q' of fibres P^h_{sx} and $P^{h'}_{sx}$ over the same point $x \in X$, there is an element $\mathrm{g} \in GL_4$ so that

$$rq' = (rq)\mathrm{g}.$$

Let $t_x \in T^*_xX$ be a cotangent vector over x. It can be expressed as

$$t_x = [rq]_M t = [rq']_M \circ \mathrm{g}^{-1}t$$

where t is some element of the standard fibre T^* of T^*X. We can write

$$\gamma_h\colon\ t_x \otimes V_x = ([rq]_M t) \otimes ([q]_V V) \to ([q]_V \circ \gamma)(t\otimes V) = \gamma_h(t(x))V_x,$$
$$\gamma_{h'}\colon\ t_x \otimes V'_x = ([rq']_M \circ \mathrm{g}^{-1})(t) \otimes ([q']_V V) \to ([q']_V \circ \gamma)(\mathrm{g}^{-1}t \otimes V) = \gamma_{h'}(t(x))V'.$$

For instance, given atlases $\{z^s_\kappa\}$ of P^h_s and $\{z'^s_\kappa\}$ of $P^{h'}_s$, the representations γ_h and $\gamma_{h'}$ read

$$\begin{aligned} t_x &= \tau_a h^a = [z^s_\kappa]_M(\tau_a t^a) = [z'^s_\kappa]_M((\mathrm{g}^{-1})^a{}_{\widetilde{b}}\tau_a t^{\widetilde{b}}) \\ &= (\mathrm{g}^{-1})^a{}_{\widetilde{a}}\tau_a h'^{\widetilde{a}} = \tau'_{\widetilde{a}} h'^{\widetilde{a}}, \\ \gamma_h(t_x) &= \tau_a\gamma^a, \\ \gamma_{h'}(t_x) &= \tau'_{\widetilde{a}}\gamma^{\widetilde{a}} = (\mathrm{g}^{-1})^a{}_{\widetilde{a}}\tau_a\gamma^{\widetilde{a}}. \end{aligned}$$

Here, we span the frames $t^h(x)$ and $t^{h'}(x)$ by different indices a and $\widetilde{a}$ since these indices are associated with different reference frames Ψ^h and $\Psi^{h'}$. Moreover, they are smeared by different world metrics $g(x)$ and $g'(x)$:

$$g(x) = [rz^s_\kappa(x)]_T\eta,$$
$$g'(x) = [rz'^s_\kappa(x)]_T\eta.$$

If $h(x) \neq h'(x)$, we have $\mathrm{g} \in GL_4\backslash L$. Then, the representations γ_h and $\gamma_{h'}$ fail to be isomorphic in the sense that there is no isomorphism ρ_V of the spinor space V such that

$$\gamma(\mathrm{g}^{-1}M \otimes \rho_V V) = \rho_V\gamma(M\otimes V).$$

For instance, the Dirac operator acting on the sum

$$\phi_h + \phi'_h$$

fails to be defined. As a consequence, Dirac fermion fields form a linear space only in the presence of a fixed tetrad gravitational field. It follows that Dirac fermion fields must be regarded only in a pair with a certain tetrad gravitational field. These pairs form the so-called fermion-gravitation complex which we aim to describe in Section 3.1.

2.3 Space-Time Structure

The geometric equivalence principle is sufficient in order that a space-time structure can be defined on a world manifold. On the other hand, one needs the global time-like γ^0-matrix operator

$$\gamma_h(h^0) = \gamma^0 \tag{2.10}$$

in order to construct the fibre spinor metric (2.2) in the bundle E^h.

In gravitational theory, a space-time structure is usually defined to be a (3+1) decomposition

$$TX = \mathbf{F} \oplus T^0X \tag{2.11}$$

of the tangent bundle over a world manifold into a 3-dimensional spatial subbundle $\mathbf{F}$ and its time-like orthocomplement T^0X.

In fibre bundle terms, we say that a world manifold X is endowed with a space-time structure if the structure group GL_4 of the principal bundle LX is reducible to its spatial rotation subgroup $SO(3)$.

Remark. By virtue of the well-known theorem [KOB], if a structure group of a principal bundle is a connected Lie group and a base of this bundle is paracompact, this structure group is reducible to its maximal compact subgroup. For instance, the structure group GL_4 of the principal bundle LX over a paracompact manifold is always reducible to its maximal compact subgroup $SO(4)$. There is the bijective correspondence between the reduced $SO(4)$-subbundles of LX and the Riemannian metrics g^R on X represented by global sections of the LX-associated Higgs bundle Σ^R with the standard fibre $GL_4/SO(4)$.

Let $L^{\mathbf{F}}X$ be a reduced $SO(3)$-subbundle of LX with a total space $P^{\mathbf{F}} \subset P$. It determines uniquely both the reduced L-subbundle L^hX of LX with the total space

$$P^h = \{p\mathbf{g}, p \in P^{\mathbf{F}}, \mathbf{g} \in L\} \tag{2.12}$$

and the reduced $SO(4)$-subbundle $L^{\mathbf{F}}X$ of LX with the total space

$$P^R = \{p\mathbf{g}, p \in P^{\mathbf{F}}, \mathbf{g} \in SO(4)\}. \tag{2.13}$$

It follows that a reduced $SO(3)$-subbundle $L^{\mathbf{F}}X$ defines uniquely a certain tetrad gravitational field h (a pseudo-Riemannian metric g) corresponding to the reduced subbundle (2.12) and a certain Riemannian metric g^R corresponding to the reduced subbundle (2.13). With respect to an atlas $\{z^{\mathbf{F}}_\kappa\}$ of LX with functions $z^{\mathbf{F}}_\kappa(x)$ taking their values in $P^{\mathbf{F}} \subset P^h$, metric functions g_κ and g^R_κ read

$$g_\kappa(x) = \eta,$$
$$g^R_\kappa(x) = \eta^{\mathbf{E}}$$

where, by $\eta^{\mathbf{E}}$, we denote the Euclidean metric. The atlas $\{z^{\mathbf{F}}_\kappa\}$ possesses $SO(3)$-valued transition functions.

With respect to a holonomic atlas, we have

$$g^R_{\mu\nu}(x) = h^a_\mu(x)h^b_\nu(x)\eta^{\mathbf{E}}_{ab} = 2h^0_\mu(x)h^0_\nu(x) - g_{\mu\nu}(x),$$
$$g^{R\mu\nu}(x) = h^\mu_a(x)h^\nu_b(x)\eta^{\mathbf{E}ab} = 2h^\mu_0(x)h^\nu_0(x) - g^{\mu\nu}(x) \tag{2.14}$$

where $h^\mu_a(x)$ are tetrad functions of a gravitational field h.

Let t^0 be a non-zero $SO(3)$-stable element of the standard fibre T^* of the cotangent bundle such that

$$\eta(t^0, t^0) = \eta^{\mathbf{E}}(t^0, t^0) = 1.$$

Given a reduced $SO(3)$-subbundle $L^{\mathbf{F}}X$, one can define the global nonvanishing 1-form (2.8)

$$h^0 = [p]_{T^*}t^0 = h^0_\mu(x)dx^\mu, \qquad p \in P^{\mathbf{F}}_x.$$

It is called the tetrad form. In particular, we may rewrite the relation (2.14) in the coordinate-free form

$$g = h^a \otimes h^b \eta_{ab},$$
$$g^R = h^a \otimes h^b \eta^{\mathbf{E}}_{ab} = 2h^0 \otimes h^0 - g.$$

The tetrad form provides us with global γ^0-operator (2.10).

Remark. There is 1:1 correspondence between the nonvanishing 1-forms ω on a manifold X and the smooth orientable distributions $\mathbf{F}$ of 1-codimensional subspaces of tangent spaces to X. This correspondence is defined by the equation

$$\mathbf{F} \rfloor \omega = 0.$$

A form ω is called a generating form of a distribution $\mathbf{F}$.

Hence, a tetrad form h^0 defines a smooth orientable distribution $\mathbf{F}$ of 3-dimensional tangent subspaces which satisfy the equation

$$\mathbf{F} \rfloor h^0 = 0.$$

Fibres $\mathbf{F}_x$ of $\mathbf{F}$ are spatial spaces with respect to the pseudo-Riemannian metric g. We therefore call $\mathbf{F}$ a space-time distribution compatible with the gravitational field g. This distribution yields the (3+1) decomposition (2.11) of the tangent bundle TX into the 3-dimensional spatial subbundle $\mathbf{F}$ and its 1-dimensional orthocomplement T^0X (with respect to g and g^R) which is generated by the nonvanishing time-like vector field

$$h_0 = h^\mu_0 \partial_\mu$$

on X. In accordance with the conventional viewpoint, such decomposition turns a world manifold X into a space-time.

In virtue of the above-mentioned theorem [KOB], if the structure group of the principal bundle LX is reducible to the Lorentz group L, this is reducible to its maximal compact subgroup $SO(3)$. Moreover, we have the following commutative diagram of reduction of structure groups of LX:

$$\begin{array}{ccc} GL_4 & \longrightarrow & SO(4) \\ \downarrow & & \downarrow \\ L & \longrightarrow & SO(3). \end{array}$$

It follows that, given a gravitational field h, the corresponding reduced subbundle L^hX always includes a reduced $SO(3)$-subbundle $L^{\mathbf{F}}X$ which determines a space-time distribution $\mathbf{F}$ compatible with the gravitational field g and the Riemannian metric g^R connected to g by the relation (2.14).

Conversely, let $\mathbf{F}$ be a smooth orientable 3-dimensional distribution with a nonvanishing generating 1-form ω. Given a Riemannian metric g^R on X and the corresponding reduced $SO(4)$-subbundle L^RX of the principal bundle LX, it is uniquely defined a reduced $SO(3)$-subbundle $L^{\mathbf{F}}X$ with a total space $P^{\mathbf{F}}$ which comprises elements $p \in P^R$ obeying the condition

$$[p]_{T^*}t^0 = \omega(x)/|\omega(x)|, \qquad x = \pi_{PX}(p),$$

where

$$|\omega|^2 = g^R(\omega, \omega).$$

In particular, we gain the following theorem [SUL, HAW] formulated in terms of space-time distributions.

Theorem. For every gravitational field h on a world manifold X, there exists an associated pair $(\mathbf{F}, g^R)$ of a g-compatible space-time distribution $\mathbf{F}$ with a generating tetrad form h^0 and a Riemannian metric g, so that

$$g^R = 2h^0 \otimes h^0 - g = h^0 \otimes h^0 + k \tag{2.15}$$

where k is a Riemannian metric in the tangent subbundle $\mathbf{F}$. Conversely, given a Riemannian metric g^R, every oriented smooth 3-dimensional distribution $\mathbf{F}$ with a generating form ω is a space-time distribution compatible with the gravitational field g given by expression (2.15) where

$$h^0 = \omega/|\omega|, \qquad |\omega|^2 = g^R(\omega, \omega) = g(\omega, \omega).$$

Moreover, we have shown that there is 1:1 correspondence between the reduced $SO(3)$-subbundles of the principal bundle LX and the triples $(g, \mathbf{F}, g^R)$ of a pseudo-Riemannian metric g, a Riemannian metric g^R, and a space-time distribution $\mathbf{F}$ which obeys relation (2.15).

A triple $(g, \mathbf{F}, g^R)$ defines uniquely a space-time structure on X. Two different triples $(g, \mathbf{F}, g^R)$ and $(g', \mathbf{F}', g'^R)$ describe the same space-time structure if the distributions $\mathbf{F}$ and $\mathbf{F}'$ coincide with each other. We say that a reference frame $\{z^{\mathbf{F}}_\kappa(x)\}$ is associated with a distribution $\mathbf{F}$ if, for any $x \in U_\kappa$,

$$([z^{\mathbf{F}}_\kappa(x)]_{T^*}t^0) \rfloor \mathbf{F} = 0.$$

Tangent vectors $t_0(x)$ associated with such a reference frame are transversal to the distribution $\mathbf{F}$. Moreover, there is a Riemannian metric g^R (and a pseudo-Riemannian metric g) such that $t_0(x)$ are orthogonal to the distribution $\mathbf{F}$.

Given $\mathbf{F}$, different reference frames $\{z^{\mathbf{F}}_\kappa\}$ and $\{z'^{\mathbf{F}}_\kappa\}$ associated with $\mathbf{F}$ differ from each other in the gauge spatial transformations $\mathbf{g}_\kappa(x) \in GL^+(3, \mathbb{R})(X)$ and in time delatations. On physical side, it means that the relative velocity of the corresponding local reference frames $[z^{\mathbf{F}}_\kappa(x)]_T(t_a)$ and $[z'^{\mathbf{F}}_\kappa(x)]_T(t_a)$ at a point $x \in X$ is equal to zero, that is, these local frames are associated with the same physical observer at this world point. We thus may conclude that, in gravitation theory, a space-time distribution $\mathbf{F}$ determines a system of local physical observers.

Given a gravitational field h, the corresponding reduced L-subbundle L^hX of the linear frame bundle LX contains different $SO(3)$-subbundles $L^{\mathbf{F}}X$ which differ from each other in principal Lorentz morphisms $l(x) \in L(X)$. Consequently, space-time distributions $\mathbf{F}$ and $\mathbf{F}'$ compatible with the same gravitational field h do as well. For their generating tetrad forms, we have the relation

$$\begin{aligned} h'^0 &= l^0{}_a(x) h^a_\mu(x) dx^\mu, \\ l^b{}_a(x) &= h'^b_\mu(x) h^\mu_a(x) \in L(X). \end{aligned} \tag{2.16}$$

A Riemannian metric g^R in a triple $(g, \mathbf{F}, g^R)$ defines a g-compatible distance function $d(x, x')$ on a world manifold X. Such a function turns X into a metric space whose locally Euclidean topology is equivalent to the manifold topology on X. Given a gravitational field g, the g-compatible Riemannian metrics and the corresponding distance functions are different for different space-time distributions $\mathbf{F}$ and $\mathbf{F}'$. It follows that physical observers associated with different distributions perceive the same world manifold as different Riemannian spaces. The well-known relativistic changes of sizes of moving bodies exemplify this phenomenon. Note however that Riemannian metrics g^R and g'^R in triples $(g, \mathbf{F}, g^R)$ and $(g, \mathbf{F}', g'^R)$ are equivalent only if principal Lorentz morphisms (2.16) between $\mathbf{F}$ and $\mathbf{F}'$ take their values in some compact neighborhood of the unit of the Lorentz group. It means that the relative velocities of local physical observers associated with $\mathbf{F}$ and $\mathbf{F}'$ must be limited by some maximal value.

One loses sight of the fact that a certain Riemannian metric and, consequently, a metric topology can be associated with a gravitational field. For instance, there were attempts of deriving a world topology directly from pseudo-Riemannian structure of a space-time (path topology, etc.) [HAW, GRO]. If a space-time obeys the strong causality condition, such topologies coincide with a familiar manifold topology of X. In general case, they however are rather extraordinary.

A world manifold X is known to admit a Riemannian metric if and only if this manifold is paracompact. If a pseudo-Riemannian metric exists on a manifold X, it is proved to be paracompact [MAR]. But the converse is not true.

World manifolds X are classified by the following characteristic classes of its tangent bundle [EGU, HIR, MAS]:

(i) the first Potryagin class $p_1(X) \in H^4(X, \mathbb{Z})$;

(ii) the Euler class $e(X) \in H^4(X, \mathbb{Z})$ if orientation of X is fixed;

(iii) the Stiefel-Whitney classes $w_i(X) \in H^i(X, \mathbb{Z}_2)$.

By $H^i(X,\mathbb{Z})$ and $H^i(X,\mathbb{Z}_2)$, we here denote the simplicial cohomology groups ($i \leq 4$).

Note that a manifold X is orientable if and only if $w_1(X)$ is the zero element of the group $H^1(X,\mathbb{Z}_2)$.

There are homomorphisms of the simplicial cohomology groups $H^*(X,\mathbb{Z})$ into the groups $H^*(X,\mathbb{R})$ of the DeRham cohomologies of exterior differential forms on X. Hence, the characteristic classes p_1 and e can be represented by the cohomology classes of the following closed characteristic forms:

$$\widehat{p}_1 = -\frac{1}{8\pi^2}\mathrm{Tr}(R \wedge R) = \frac{1}{8\pi^2}R^{ab} \wedge R_{ab},$$
$$\widehat{e} = \frac{1}{32\pi^2}\varepsilon_{abcd}R^{ab} \wedge R^{cd}. \tag{2.17}$$

Here, the curvature 2-form R of a linear connection Γ on TX is assumed to take its values in Lie algebras $\mathfrak{so}$(4-k,k):

$$R = \frac{1}{2}R^{ab}I_{ab}, \qquad I_{ab} = -I_{ba},$$

where generators I_{ab} act on $\mathbb{R}^4$.

If X is a compact manifold, we have the Pontryagin number

$$P_1 = \int_X \widehat{p}_1$$

and the Euler characteristic

$$\chi = \int_X \widehat{e}$$

of the manifold X.

Metric gravitational fields exist on noncompact manifolds and on compact manifolds whose Euler characteristic is equal to zero. To define a tetrad gravitational field and a space-time structure, one must require a world manifold be spatially oriented (when each constituent of the decomposition (2.11) is oriented).

Spinor bundles E with the structure group $L_s = SL(2,\mathbb{C})$ are classified by the Chern classes $c_i(E) \in H^{2i}(X,\mathbb{Z})$. Since the group L_s is reducible to its maximal compact subgroup $SU(2)$, the class $c_1(E)$ is proved to be the zero element of the group $H^2(X,\mathbb{Z})$. Chern classes are represented by the cohomology classes of the characteristic forms

$$\widehat{c}_1 = \frac{i}{2\pi}\mathrm{Tr}\,\mathcal{F} = 0,$$
$$\widehat{c}_2 = \frac{1}{8\pi^2}\mathrm{Tr}(\mathcal{F}\wedge\mathcal{F}) = -\frac{1}{32\pi^2}\mathcal{F}^{ab}\wedge\mathcal{F}_{ab}. \tag{2.18}$$

Here, $\mathcal{F}$ is the curvature 2-form of a principal connection A_s on E:

$$\mathcal{F} = \frac{1}{2}\mathcal{F}^{ab}I_{ab}$$

where generators I_{ab} act on the 2-dimensional complex space.

Because of the inclusion

$$GL_4 \to GL(4,\mathbb{C}),$$

the tangent bundle TX can be regarded as a $GL(4,\mathbb{C})$-bundle T^cX. In particular, we have

$$p_1(X) = -c_2(T^cX).$$

If a spinor bundle $E \cong E^h$ is associated with the tangent bundle $TX \cong M^hX$ (Section 2.2), the bundle T^cX is associated with the tensor product $E \otimes E^*$. It follows that

$$p_1(X) = -c_2(T^cX) = -4c_2(E),$$
$$w_2(X) = 0.$$

We can reproduce the first relation for the characteristic forms (2.17) and (2.18) if a world connection $\Gamma = \Gamma^h$ on LX is the Lorentz connection induced by the spinor connection on E, that is, if $\mathcal{F}^{ab} = R^{ab}$.

A world manifold thus must satisfy certain topological conditions in order to admit a gravitational field, a spinor structure and a space-time structure. If X is not compact, its tangent bundle must be trivial. For a compact X, its Euler characteristic and the Stiefel-Whitney classes $w_{1,2}$ must be zero and its first Pontryagin number must be multiple of 48 [GER, WHI]. A compact manifold however fails to be provided with a causal space-time structure.

A 1-codimensional distribution $\mathbf{F}$ is called an integrable distribution if its generating form ω obeys the equation

$$\omega \wedge d\omega = 0.$$

In this case, fibres of the corresponding (3+1) decomposition (2.11) are tangent to leaves of some 1-codimensional foliation of spatial hypersurfaces of a world manifold X. An integrable space-time distribution (a space-time foliation) is called causal if its generating form can be exact, that is,

$$\omega = df$$

where f is some real function on X which has no critical points where $df = 0$ [LAW]. This notion of causality coincides with the one of stable causality by Hawking [HAW]. Leaves of a causal foliation are level surfaces of its generating function f. No curve transversal to leaves of a causal foliation intersects each leave more than once. It follows that a causal space-time manifold is no compact.

Thus, if we do not concern gravitation singularities, we can restrict our consideration to the world manifolds possessing the trivial tangent bundles.

Appendix. Space-Time Singularities

Various criterion of gravitational singularities have been suggested [HAW, CAN 1988, FUC]. In view of the above-mentioned correspondence between the space-time distributions and the gravitational fields on a world manifold X, some type of gravitational singularities can be indicated by singularities of space-time foliations [SAR 1986]. In particular, caustic singularities of space-time foliations exemplify superposition of classical gravitational fields which we shall discuss in Section 3.2.

We say that a gravitational field g on X is free from singularities if there exists an associated pair of a complete Riemannian metric g^R and a causal space-time foliation $\mathbf{F}$ with the generating form h^0 such that $g^R(\nabla h^0, \nabla h^0)$ is bounded on X. This condition guaranties that, being complete with respect to g^R, a space-time satisfies the well-known b-completeness condition [CAN 1984].

One can distinguish several types of gravitation singularities in accordance to this criterion. In this Appendix, we examine gravitation singularities characterized by singularities of space-time distributions. The distribution singularities can be described locally (in the germ form) as singularities of a causal foliation with a generating function f. There are two types of these singularities.

(i) A single-valued generating function f has critical points where $df = 0$. It generates the Haefliger structure (the singular foliation) of its level sets on X. These level sets change their topology at critical points of f. Gravitation singularities of this type are the scalar curvature singularities in accordance to the classification in ref. [ELL].

(ii) A generating function f is a multiple-valued function on X. The leaves of the foliation $\mathbf{F}$ defined on the domain where f is a single-valued begin to intersect each other at branch points of f. Branch points of f where the foliation is destroyed form a caustic. To describe foliation singularities of this type, one can lift the space-time foliation $\mathbf{F}$ into the total space of the cotangent bundle T^*X, then extend this lifted foliation over the singular points, and project the extended foliation onto the base X. Singularities of $\mathbf{F}$ can be described as singularities of this projection.

In gravitation theory, a geometrical locus of focal and conjugate points is called a caustic by analogy with geometrical optics [WAR, ROS]. We follow the general mathematical notion of caustics as singularities of the Lagrange maps [ARN, FRI]. Each caustic can be brought locally (in germ terms) into the following standard form.

Let a space $\mathbb{R}^{2n}$ be endowed with the coordinates $\{x^\mu, P_\mu\}$. Let us consider the Liouville form

$$\alpha = P_\mu dx^\mu \tag{2.19}$$

on $\mathbb{R}^{2n}$ and a submanifold N of $\mathbb{R}^{2n}$ such that

$$d\alpha(N) = 0,$$

that is, being restricted onto N, the form α is exact:

$$\alpha(N) = dz(N).$$

Such a manifold of maximal dimension n is called a Lagrange submanifold. A Lagrange submanifold can be defined by a generating function $S(x^i, P_j)$ of n variables $(x^i, P_j\colon\ i \in I,\ j \in J)$ (where (I, J) is some partition of the set $(1,\ldots,n)$. It is given by the relations

$$x^j = -\frac{\partial S}{\partial P_j}, \qquad P_i = \frac{\partial S}{\partial x^i}.$$

Let us consider the projection

$$\pi\colon\ (x^\mu, P_\mu) \to (x^\mu)$$

of $\mathbb{R}^{2n}$ onto $\mathbb{R}^n$. Being restricted to the Lagrange submanifold

$$\pi_N\colon\ (x^i, P_j) \to \left(x^i, x^j = -\frac{\partial S}{\partial P_j}\right),$$

this projection is called the Lagrange map. A caustic is defined to be the set of critical points of a Lagrange map, i.e., the points where the matrix

$$\partial^2 S / \partial P_i \partial P_j$$

is singular.

For instance, a caustic on manifolds is defined as follows. Let the cotangent bundle T^*X be provided with the induced coordinates $(x^\mu, P_\mu, \dot{x}_\mu)$. The Liouville form (2.19) defines n-dimensional Lagrange submanifolds of T^*X. Singular points of projection of such a Lagrange submanifold onto the base X form a caustic.

Let us note that a geometrical locus of focal and conjugate points of Riemannian and time-like pseudo-Riemannian geodesics also is a caustic in accordance to the Arnol'd definition [SAR 1986].

Our definition of foliation caustics is based on the following proposition.

Proposition. For any foliation of level surfaces $\mathbf{F}$ of a manifold X, there is a foliation $\mathbf{F}'$ of some Lagrange submanifold of T^*X such that $\mathbf{F}$ is the image of $\mathbf{F}'$ under the Lagrange map.

Outline of proof: Let f be a generating function of the foliation $\mathbf{F}$. We define the embedding

$$\gamma\colon\ (x^\mu) \to (x^\mu, P_\mu = \frac{\partial f}{\partial x^\mu})$$

of X into T^*X. Its image is a Lagrange submanifold of $T^*(X)$. Let $\mathbf{F}'$ be the induced foliation $\pi^*_{\gamma(X)}F$ of $\gamma(X)$ where $\pi_{\gamma(X)}$ is the Lagrange map

$$\pi_{\gamma(X)}\colon \gamma(X) \to X.$$

Since γ and $\pi_{\gamma(X)}$ are diffeomorphisms between X and $\gamma(X)$ ($\pi_{\gamma(X)} \circ \gamma = \mathrm{id}\, X$), the foliation $\mathbf{F}$ on X can be represented as the image of the foliation $\mathbf{F}'$ on $\gamma(X)$ under the Lagrange map $\pi_{\gamma(X)}$.

For instance, let $N \subset T^*X$ be the Lagrange submanifold generated locally by a function $S(x^i, P_j)$, and let $\mathbf{F}'$ be the foliation of level surfaces of the function

$$f'(x^i, P_j) = S - P_j \frac{\partial S}{\partial P_j}$$

on the Lagrange submanifold N. The image $\pi_N(\mathbf{F}')$ of $\mathbf{F}'$ under the Lagrange map π_N is a foliation of the image $\pi_N(U)$ of the domain $U \subset N$ where this Lagrange map has no critical points. This foliation is destroyed at caustic points of the Lagrange map π_N.

There are six classes A_2, A_3, A_4 D_4, A , D of stable caustics on a 4-dimensional manifold. For example, the canonical generating function of caustic A_3 takes the form

$$S = -P_0^4 + x^1 P_0^2,$$

and the corresponding Lagrange manifold N is given by equations

$$x^0 = 4P_0^3 - 2x^1 P_0, \qquad P_1 = P_0^2.$$

The Lagrange map then reads

$$x^0 = 4P_0^3 - 2x^1 P_0. \tag{2.20}$$

The caustic set where

$$\partial^2 S / \partial P_0^2 = 0$$

consists of the points

$$x^1 = 6P_0^2,$$

and its Lagrange image on X contains the points

$$(x^0)^2 = \frac{8}{27}(x^1)\ .$$

The generating function of the foliation $\mathbf{F}'$ on the Lagrange manifold takes the form

$$f' = 3P_0^4 - x^1 P_0^2.$$

Then, on X, the generating function of the Lagrange image of $\mathbf{F}'$ reads

$$f(x^0, x^1) = f'(x^1, P_0(x^0, x^1))$$

where the function $P_0(x^0, x^1)$ is defined by equation (2.20). The function $P_0(x^0, x^1)$ and the generating function f become three-valued functions at caustic points. The A_3-germ of foliation caustics thus is characterized by the behavior of the component ω_0 of the foliation generating form which is tripled at caustic points.

Caustic singularities have the following feature. There are domains of a space-time where not nearest, but the far separated leaves begin to intersect each other. Therefore, a space-time foliation can be locally prolonged over the caustic points, whereas global prolongation of this foliation is impossible.

For example, let $f(u, v)$ be a real function on $\mathbb{R}^2$ which obeys the equation

$$f^3(u, v) - 3uf(u, v) - 2v = 0$$

where u, v are coordinates on $\mathbb{R}^2$. This function is the singled-valued one

$$f_+ = \left[v + (v^2 - u^3)^{1/2}\right]^{1/3} + \left[v - (v^2 - u^3)^{1/2}\right]^{1/3}$$

on the domain $U = (u, v\colon v^2 > u^3)$, and it is the three-valued function

$$f_{0,1,2} = 2u^{1/2} \cos(\frac{1}{3}(\phi + 2\pi n)),$$

$$\phi = \arccos(vu^{-3/2}), \qquad n = 0, 1, 2,$$

at points $v^2 < u^3$. Let $\mathbf{F}$ be the foliation

$$\mathbf{F}_c = \{u, v\colon f_+(u, v) = c = \mathrm{const}\}$$

on $U \subset \mathbb{R}^2$. Its leaves $\mathbf{F}_c$ are the lines

$$2v = c^3 - 3uc, \qquad -\infty < c < +\infty.$$

This foliation has the caustic singularity at the branch points $v^2 = u^3$ of the function f. Moreover, $u = v = 0$ is the A_3-caustic point, whereas the other ones $v^2 = u^3 \neq 0$ are points of the A_2-caustic. The leaves $\mathbf{F}_c$, $c > \varepsilon > 0$, can be prolonged over the caustic curve $v = u^{3/2}$ onto the domain $0 < v < u^{3/2}$ where they can be described as leaves of the foliation

$$f_0(u, v) = \mathrm{const}.$$

These leaves however begin to intersect each other when $v < 0$, although the nearest leaves intersect each other only on the caustic curve $v = -u^{3/2}$. Note that the leaves $\mathbf{F}_{c>0}$ begin to intersect the leaves $\mathbf{F}_{c<0}$ on the caustic curve $v = u^{3/2}$.

Caustic singularities however are not reduced to the locally extensible singularities [ELL]. For instance, the A_2-caustic points $u^2 = v^3 \neq 0$ of the above-mentioned foliation caustic are locally extensible singularity points, whereas the A_3-caustic point $u = v = 0$ is not locally extensible.

Chapter 3

THE HIGGS FEATURE OF CLASSICAL GRAVITY

For the first time, the conception of a graviton as a Goldstone particle corresponding to violation of Lorentz symmetries in a curved space-time was expressed in mid 60s in connection with matching cosmological and vacuum asymmetries by Heisenberg and Ivanenko.

This idea was revived in the framework of the formalism of nonlinear realizations [OGI, ISH]. The Lorentz group is the Cartan subgroup of GL_4, and the induced representations $L \uparrow GL_4$ have been constructed [NEE 1979]. In this approach, geometric aspects of gravity however were ignored.

In fibre bundle terms, the fact that a pseudo-Riemannian metric is *sui generis* a Higgs field has been pointed out by Trautman [TRA 1979] and by us [SAR 1980]. Our description of the spontaneous breakdown of world symmetries is based on the geometric equivalence principle and on the existence of Dirac fermion fields [IVA 1983, SAR 1991]. The Higgs field feature of gravity issues from the fact that different tetrad gravitational fields h and h' define the nonisomorphic representations γ_h and $\gamma_{h'}$ which we have constructed above. It follows that Dirac fermion fields and gravitational fields form the peculiar *fermion-gravitation complex* and that gravitational fields do not admit geometric deviations.

3.1 Fermion-Gravitation Complex

Since Dirac fermion fields must be regarded only in a pair with a certain gravitational field, the totality of these pairs fails to be formalized by the bundle product $\Sigma \times E$ of the Higgs bundle Σ and some spinor bundle E, but forms the so-called fermion-gravitation complex [NEE 1979]. To describe this complex, we follow the construction developed in Section 1.4.

The total space P of the principal bundle LX is the total space of the principal bundle P^L with the base

$$\Sigma = P/L$$

and the structure group L.

It seems natural to assume that a world manifold X is no compact and that the principal bundle LX then is trivial (Section 2.3). Since the bundle (2.3) is trivial,

the principal bundle P^L is also trivial. Therefore, there exists the lift of P^L to a trivial principal spinor bundle P^L_s over Σ with the structure group L_s such that

$$P^L = rP^L_s = P^L_s/\mathbb{Z}_2, \qquad P^L_s/L_s = \Sigma.$$

In particular, given a global section h of the Higgs bundle Σ, the L_s-principal bundle P^h_s is the portion of P^L_s over $h(X) \subset \Sigma$.

Let us provide the principal bundle LX with a holonomic atlas Ψ^T and the bundle P^L_s and P^L with some associated atlases $\{z^L_s\}$ and

$$\{z^L = r \circ z^L_s\}.$$

We can choose atlases $\{z_s\}$ and $\{z^L\}$ with the identity transition functions. At the same time, a holonomic atlas fails to consist of one chart in general.

We endow the Higgs bundle Σ with the bundle coordinates $(x^\lambda, \sigma^\lambda_a)$ where $\sigma^\lambda_\alpha(\sigma)$ are matrix components of the transformation

$$(\psi^T_\kappa \circ z^L)(\sigma)\colon\ t_a \to \sigma^\lambda_a t_\lambda$$

of the fixed basis for the standard fibre T of the tangent bundle TX. In particular, we have

$$z^h(x) = z^L(h(x)),$$
$$\sigma^\mu_a(h(x)) = h^\mu_a(x)$$

where $h^\mu_a(x)$ are tetrad functions (2.4).

Let (E^L, Σ, F, L) be a spinor bundle associated with P^L_s. Following Section 1.4, we consider the composite bundle

$$\pi_{EX} = \pi_{\Sigma X} \circ \pi_{E\Sigma}\colon\ \widetilde{E} = E^L \to \Sigma \to X. \tag{3.1}$$

The bundles E^L and $\widetilde{E}$ are trivial. The composite bundle $\widetilde{E}$ is provided with the bundle coordinates of the bundle E^L:

$$E^L = \widetilde{E} \ni y = (x^\lambda, \sigma^\mu_a, v_A).$$

Given a global section h of the Higgs bundle Σ, the spinor bundle E^h associated with the principal bundle P^h_s is the portion of the bundle E^h over $h(X) \subset \Sigma$. It follows that every global section $\widetilde{\phi}$ of $\widetilde{E}$ can be represented by a pair $(\phi_h(x), h(x))$ of fields which depend on world coordinates and describe a fermion field $\phi_h(x)$ in the presence of the tetrad gravitational field

$$h(x) = (\pi_{E\Sigma} \circ \widetilde{\phi})(x).$$

Remark. The trivial bundle $\widetilde{E}$ is isomorphic to the bundle product

$$\widetilde{E} = E \underset{X}{\times} \Sigma$$

where E is some trivial spinor bundle over X, e.g., $E = E^h$. This splitting however fails to be compatible with the γ-matrix representations of cotangent vectors because the representations γ_h and $\gamma_{h'}$ on the same space E_x in different products $E_x \times h(x)$ and $E_x \times h'(x)$ are not isomorphic.

The bundle $\widetilde{E}$ is not associated with a principal bundle and to construct a connection on $\widetilde{E}$, we use the jet manifold morphism

$$\xi\colon J^1\Sigma \underset{\Sigma}{\times} J^1E^L \to J^1\widetilde{E}.$$

Let us provide the jet manifolds $J^1\Sigma$, J^1E^L and $J^1\widetilde{E}$ with the adapted coordinates

$$\begin{aligned} &J^1\Sigma \ni s = (x^\lambda, \sigma^\mu_a, \sigma^\mu_{a\lambda}),\\ &J^1E^L \ni w = (x^\lambda, \sigma^\mu_a, v_A, v_{A\lambda}, v^a_{A\lambda}),\\ &J^1\widetilde{E} \ni q = (x^\lambda, \sigma^\mu_a, v_A, \sigma^\mu_{a\lambda}, \widetilde{v}_{A\lambda}). \end{aligned}$$

In this coordinates, the morphism ξ reads

$$\widetilde{v}_{A\lambda} = v^a_{A\mu}\sigma^\mu_{a\lambda} + v_{A\lambda}.$$

Given a world connection Γ on the bundle LX and a principal spinor connection A^L on the bundle P^L_s, we can induce a general connection $\widetilde{\Gamma}$ on the bundle $\widetilde{E}$:

$$\begin{aligned} &\Gamma = dx^\lambda \otimes (\partial_\lambda - \Gamma^\mu{}_{\nu\lambda}(x)\sigma^\nu_a\partial^a_\mu),\\ &A^L = dx^\lambda \otimes (\partial_\lambda + A_\lambda{}^B{}_A(\sigma)v_B\partial^A) + d\sigma^\lambda_a \otimes (\partial^a_\lambda + A^{aB}_\lambda{}_A(\sigma)v_B\partial^A),\\ &\widetilde{\Gamma} = dx^\lambda \otimes (\partial_\lambda + \widetilde{\Gamma}_\lambda{}^\mu_a\partial^a_\mu + \widetilde{\Gamma}_{\lambda A}\partial^A)\\ &\quad = dx^\lambda \otimes [\partial_\lambda - \Gamma^\mu{}_{\nu\lambda}(x)\sigma^\nu_a\partial^a_\mu + (-\Gamma^\mu{}_{\nu\lambda}(x)\sigma^\nu_a A^{aB}_\mu{}_A(\sigma) + A_\lambda{}^B{}_A(\sigma))v_B\partial^A]. \end{aligned}$$

Let us fix a gravitational field h and consider sections $\widetilde{\phi}$ of $\widetilde{E}$ represented by the pairs (ϕ_h, h). The covariant differential (0.49) of such sections then is given by the coordinate expression

$$\begin{aligned} \widetilde{D}_\lambda h^\mu_a(x) &= \partial_\lambda h^\mu_a(x) + \Gamma^\mu{}_{\nu\lambda}(x)h^\nu_a(x),\\ \widetilde{D}_\lambda\phi_A(x) &= \partial_\lambda\phi_A(x) - [-\Gamma^\mu{}_{\nu\lambda}(x)h^\nu_a(x)A^{aB}_\mu{}_A(x^\lambda, h^\nu_a(x))\\ &\quad + A_\lambda{}^B{}_A(x^\lambda, h^\nu_a(x))]\phi_B(x). \end{aligned} \tag{3.2}$$

The Higgs field contribution term (1.68) in expression (3.2) then reads

$$h^\nu_a(x)\Gamma^\mu{}_{\nu\lambda}(x)A^{aB}_\mu{}_A(x^\lambda, h^\nu_a(x))\phi_B(x).$$

The covariant derivatives (3.2) of a fermion field ϕ_h are reduced to the familiar ones if a connection A^L on the bundle P^L_s is a pull-back connection.

One can generalize the Dirac operator (2.9) to the one on the jet manifold $J^1\widetilde{E}$.

Let E^L_M be the Minkowski space bundle associated with the L-principal bundle P^L. Using the morphism (2.1), we can define the bundle morphism

$$\gamma_L\colon E^L_M \underset{\Sigma}{\times} E^L \to E^L$$

by analogy with the morphism γ_E.

Since the bundle P^L is trivial, we have the bundle isomorphism over Σ:

$$\zeta\colon T^*X \to E^L_M \tag{3.3}$$

where T^*X denote the pull-back $H^*\Sigma$ of the cotangent bundle T^*X by $\pi_{\Sigma X}$ and E^L_M is regarded as the $\mathbb{R}^4$ bundle, not the Minkowski space bundle. It means that, given an atlas $\{z^L\}$ of E^L_M and the pull-back holonomic atlas Ψ^T of the pull-back bundle T^*X, isomorphism (3.3) can be defined by the fibre-to-fibre morphism

$$\sigma^\mu_a\partial_\mu \to t_a(\sigma)$$

where $t_a(\sigma)$ and ∂_μ are the bases associated with atlases $\{z^L\}$ and Ψ^T for the fibres of the bundles E^L_M and T^*X at a point $\sigma \in \Sigma$. Being restricted to the submanifold $h(X) \subset \Sigma$ for some section h of Σ, this isomorphism (3.3) is reduced to the isomorphism of the cotangent bundle T^*X to the Minkowski space bundle M^hX.

The composite of the morphisms γ_L and ξ results in the γ-matrix representation morphism

$$\gamma_\Sigma = \gamma_L \circ \zeta\colon T^*X \underset{\Sigma}{\times} E^L \to E^L.$$

Being restricted to the submanifold $h(X) \subset \Sigma$ for some section h of Σ, the morphism γ_Σ is reduced to the morphism γ_h (2.7).

Since the bundle $\widetilde{E}$ is trivial, we can write

$$V\widetilde{E} = VE^L \underset{\widetilde{E}}{\times} V\Sigma$$

where $V\Sigma$ denotes the pull-back $\pi^*_{E\Sigma}(V\Sigma)$ of the vertical bundle $V\Sigma$ over Σ. Given a connection $\widetilde{\Gamma}$ on the bundle $\widetilde{E}$, the covariant differential (0.44) then defines the morphism

$$D\colon J^1\widetilde{E} \to T^*X \underset{\widetilde{E}}{\otimes} V\widetilde{E} = \left(T^*X \underset{\widetilde{E}}{\otimes} VE^L\right) \underset{\widetilde{E}}{\times} \left(T^*X \underset{\widetilde{E}}{\otimes} V\Sigma\right)$$

such that

$$D_\phi = \mathrm{pr}_1 \circ D\colon J^1\widetilde{E} \to \left(T^*X \underset{\widetilde{E}}{\otimes} VE^L\right)$$

is the covariant differential (3.2). Using the canonical vertical splitting of the vector bundle VE^L, we then can define the generalized Dirac operator

$$L_\Sigma = \gamma_\Sigma \circ D_\phi\colon\ J^1\widetilde{E} \to VE^L = \mathrm{pr}_1(V\widetilde{E}),$$
$$L_\Sigma = \sigma^\lambda_a \gamma^{aB}{}_A(\widetilde{v}_{B\lambda} - \widetilde{\Gamma}_{\lambda B})\partial^A.$$

For each section $\widetilde{\phi} = (\phi_h, h)$ of the bundle $\widetilde{E}$, this operator is reduced to the Dirac operator on sections of the bundle E^h, but in the presence of the generalized connection (3.2):

$$L_\Sigma \phi_A = h^\lambda_a(x)\gamma^{aB}{}_A \widetilde{D}_\lambda \phi_B(x).$$

To construct a connection $\widetilde{\Gamma}$ on the bundle $\widetilde{E}$, we use a principal connection A^L on the principal bundle P^L_s over the Higgs manifold Σ. Therefore, a total Lagrangian for the fermion-gravitation complex must include a Lagrangian of principal connections A^L.

Remark. Describing fields on the Higgs manifold Σ, one faces the problem of a fibre metric in $T^*\Sigma$. We can suggest the DeWitt fibre metric given by the expression

$$G_W(\omega,\omega') = \sigma^\mu_a \sigma^\nu_b \eta^{ab} X_\mu X'_\nu + G^{\mu\nu}_{ab}(\sigma)\Sigma^a_\mu \Sigma'^b_\nu,$$
$$G^{\mu\nu}_{ab} = \frac{1}{2}\det|\sigma^\mu_a|(\sigma^\mu_p\sigma^\nu_q\eta^{pq}\eta_{ab} + \sigma^\mu_b\sigma^\nu_a - \sigma^\mu_a\sigma^\nu_b),$$
$$\omega = X_\mu dx^\mu + \Sigma^a_\mu d\sigma^\mu_a, \qquad \omega' = X'_\mu dx^\mu + \Sigma'^a_\mu d\sigma^\mu_a.$$

This metric however is degenerate because of the degenerate matrix $G^{\mu\nu}_{ab}$. Another fibre metric in $T^*\Sigma$ can be constructed by analogy with the Schmidt-Marathe metric in the linear frame bundle LX [MAR, CAN 1984]. Given a world connection Γ on LX, this metric reads

$$G_{SM}(\omega,\omega') = \sigma^\mu_a\sigma^\nu_b[X_\mu X'_\nu \eta^{ab}$$
$$+ (\Sigma^a_\mu - \Gamma^\gamma_{\mu\beta}(x)\sigma^\beta_q\eta^{qa}X_\gamma)(\Sigma'^b_\nu - \Gamma^\gamma_{\nu\beta}(x)\sigma^\beta_q\eta^{qb}X'_\gamma)].$$

In this expression, a connection Γ makes the sense of some background connection. Using the metrics G_W and G_{SM}, one can construct, e.g., the generalized D'Alamber operators acting on fields on the Higgs bundle Σ:

$$\Box_W = G_W(\theta_\Sigma, \theta_\Sigma),$$
$$\Box_{SM} = G_{SM}(\theta_\Sigma, \theta_\Sigma),$$
$$\theta_\Sigma = \frac{\partial}{\partial x^\mu}dx^\mu + \frac{\partial}{\partial \sigma^\mu_a}d\sigma^\mu_a$$

On pull-back fields, the operator $\Box_{SM}$ however fails to reduce to the familiar D'Alamber operator $\Box$.

In gauge gravitation models, independent gravitational variables are the pairs of a tetrad gravitational field h and a gauge gravitational potential represented by a *principal connection A on the reduced subbundle P^h of the principal bundle* LX [OBU]. If one considers fermion fields, connections A are induced by principal spinor connections which describe gauge potentials of interaction between fermion fields. A world connection Γ on LX is assumed to be the extensions of a connection A to LX. Given a holonomic atlas Ψ^T and an atlas Ψ^L of the principal bundle LX, a tetrad field is represented by tetrad functions (2.4) and a connection A by coefficients $A^{ab}{}_\mu(x)$ of the local connection form (1.3) where

$$A^{ab}{}_\mu(x) = -A^{ba}{}_\mu(x) = A^m_\mu(x)(I_m)^{ab} = -\tfrac{1}{2}A^{cd}{}_\mu(x)(I_{cd})^{ab},$$
$$(I_{cd})^{ab} = \delta^b_c\delta^a_d - \delta^a_c\delta^b_d$$

In the framework of this so-called first order formalism, gravitational Lagrangians are constructed by means of the curvature

$$\mathcal{F}^{ab}{}_{\mu\nu} = \partial_\mu A^{ab}{}_\nu - \partial_\nu A^{ab}{}_\mu + A^{ac}{}_\mu A^b{}_{c\nu} - A^{ac}{}_\nu A^b{}_{c\mu}$$

and the torsion

$$\Omega^a{}_{\mu\nu} = \frac{1}{2}(\partial_\nu h^a_\mu - \partial_\mu h^a_\nu + A^a{}_{b\nu}h^b_\mu - A^a{}_{b\mu}h^b_\nu).$$

Many authors however believe that the classical gravity is characterized only by the Hilbert-Einstein Lagrangian density

$$\mathcal{L}_{HE} = -\frac{1}{2\kappa}\mathcal{F}^{ab}{}_{\mu\nu}h^\mu_a h^\nu_b h^{-1}, \qquad h = \det|h^\mu_a|. \tag{3.4}$$

To reproduce familiar results in the framework of the fermion-gravitational complex, we restrict our consideration to the pull-back principal connections A^L on P^L_s and to those principal connections on LX which are induced by these pull-back connections.

Let C^L be the *connection bundle of principal connections on P^L_s* and $H^*\Sigma$ be the horizontal cotangent subbundle of the cotangent bundle $T^*\Sigma$. The configuration space of pull-back principal spinor connections is the subspace

$$\widetilde{C} = C^L_+ \underset{C^L}{\oplus} (\overset{2}{\wedge} H^*\Sigma \otimes V^L P^L_s) \tag{3.5}$$

of the configuration space

$$J^1C^L = C^L_+ \underset{C^L}{\oplus} (\overset{2}{\wedge} T^*\Sigma \otimes V^L P^L_s).$$

The total configuration space of the fermion-gravitation complex is then reduced to the product

$$J^1\widetilde{E} \underset{\Sigma}{\times} \widetilde{C}.$$

For the sake of simplicity, we here examine gravity without matter. Its configuration space is

$$J^1\Sigma \underset{\Sigma}{\times} \widetilde{C}. \tag{3.6}$$

In expression (3.5), given the affine bundle

$$C^L_+ = J^2 P^L_s / L_s$$

modelled on the vector bundle $\overset{2}{\vee} T^*\Sigma \otimes V^L P^L_s$, we can restrict ourselves to one of its affine subbundles $\widetilde{C}_+$ modelled on the vector bundle $\overset{2}{\vee} H^*\Sigma \otimes V^L P^L_s$. Points of $C^L_+ \backslash \widetilde{C}_+$ are conjugated to points of $\widetilde{C}_+$ by elements of the gauge group $L_s(\Sigma)$. If a trivial atlas $\{z^L_s\}$ of the bundle P^L_s and the associated coordinates $(x^\mu, \sigma^\lambda_a)$ on Σ and $(x^\mu, \sigma^\lambda_a, k^m_\mu, k^m{}^a_\lambda)$ on C^L are fixed, it seems natural to choose the subbundle $\widetilde{C}_+$ defined by the coordinate relations

$$k^m{}^a_\lambda = 0 \qquad s^m{}^a_{\lambda\mu} = 0, \qquad s^m{}^{ab}_{\lambda\mu} = 0$$

where we use the notations (1.8). The configuration space (3.6) hence can be reduced to its subspace

$$J^1\Sigma \underset{\Sigma}{\times} (\widetilde{C}_+ \underset{C^L}{\oplus} (\overset{2}{\wedge} H^*\Sigma \otimes V^L P^L_s) \tag{3.7}$$

given by the adapted coordinates

$$\begin{aligned}
&(x^\mu, \sigma^\lambda_a, k^{ab}{}_\lambda = -k^{ba}{}_\lambda, \sigma^\lambda_{a\mu}, s^{ab}{}_{\mu\lambda}, F^{ab}{}_{\mu\lambda}),\\
&s^{ab}{}_{\mu\lambda} = k^{ab}{}_{\mu\lambda} + k^{ab}{}_{\lambda\mu},\\
&F^{ab}{}_{\mu\lambda} = k^{ab}{}_{\lambda\mu} - k^{ab}{}_{\mu\lambda} + k^{ac}{}_\mu k^b{}_{c\lambda} - k^{ac}{}_\lambda k^b{}_{c\mu}.
\end{aligned} \tag{3.8}$$

Here, we replace the index m of the fibre coordinates k^m on P^L at base points $(x^\mu, \sigma^\lambda_a)$ by the pair ab of indices of the coordinates σ^λ_a.

In the coordinates (3.8) on the configuration space (3.7), the Lagrangian density (3.4) of the classical gravity reads

$$\mathcal{L}_{HE} = -\frac{1}{2\kappa}\mathcal{F}^{ab}{}_{\mu\lambda}\sigma^\mu_a\sigma^\lambda_b\sigma^{-1}, \qquad \sigma = \det|\sigma^\mu_a|. \tag{3.9}$$

This Lagrangian density describes the dynamic system with the following primary constraints:

$$\begin{aligned}
\pi^{\mu\lambda}{}_{ab} &= \frac{\partial\mathcal{L}}{\partial k^{ab}{}_{\lambda\mu}} = \frac{1}{2\kappa}\sigma^{-1}(\sigma^\lambda_a\sigma^\mu_b - \sigma^\mu_a\sigma^\lambda_b),\\
\pi^\mu{}^a_\lambda &= \frac{\partial\mathcal{L}}{\partial\sigma^\lambda_{a\mu}} = 0.
\end{aligned} \tag{3.10}$$

Therefore, we aim to apply the multimomentum Hamiltonian formalism to the gauge gravitation theory. In comparison with the familiar Hamiltonian formalism [NIC], it does not need the preliminary (3+1) decomposition of tensor bundles over a world manifold.

For the sake of simplicity, we restrict our consideration to the local constructions on a standard coordinate chart of the corresponding Legendre manifold Π:

$$(x^\mu, \sigma^\lambda_a, k^{ab}{}_\lambda = -k^{ba}{}_\lambda, q^\mu{}^a_\lambda, p^{\mu\lambda}{}_{ab} = -p^{\lambda\mu}{}_{ab}). \tag{3.11}$$

Let us fix a pull-back atlas $\{z^L\}$ of the principal bundle P^L_s which corresponds to some isomorphism of P^L_s to the pull-back

$$\widetilde{P}_s = \widetilde{P}^L_s \underset{X}{\times} \Sigma = (\pi_{\Sigma X})^* \widetilde{P}^L_s$$

of some trivial L_s-principal bundle $\widetilde{P}^L_s$ over X by the projection $\pi_{\Sigma X}$. Then, the coordinates $(x^\mu, \sigma^\lambda_a, q^\mu{}^a_\lambda)$ are local coordinates on the bundle

$$\overset{n}{\wedge} T^*X \otimes TX \otimes T^*X \underset{\Sigma}{\otimes} M\Sigma$$

where $M\Sigma$ is the pull-back of the Minkowski space bundle MX associated with $\widetilde{P}^L_s$. The coordinates $(x^\mu, \sigma^\lambda_a, k^{ab}{}_\lambda, p^{\mu\lambda}{}_{ab})$ are the local standard coordinates on the pull-back of the bundle Π^1_A by projection $\pi_{\Sigma X}$ where Π_A is the Legendre manifold of principal connections on the bundle $\widetilde{P}^L_s$.

The image of the Legendre morphism $\widehat{L}_{HE}$ is given by the coordinate relations

$$p^{\mu\lambda}{}_{ab} = \pi^{\mu\lambda}{}_{ab}, \qquad q^\mu{}^a_\lambda = \pi^\mu{}^a_\lambda = 0$$

where $\pi^{\mu\lambda}{}_{ab}$ and $\pi^\mu{}^a_\lambda$ are the quantities (3.10).

In the coordinates (3.11), the multimomentum Hamiltonian form associated with the Lagrangian density (3.9) reads

$$H_{HE} = (q^\mu{}^a_\lambda d\sigma^\lambda_a + p^{\mu\lambda}{}_{ab} dk^{ab}{}_\lambda) \wedge \omega_\mu - \mathcal{H}\omega, \tag{3.12}$$

$$\mathcal{H} = -\frac{1}{2}p^{\mu\lambda}{}_{ab}(k^{ac}{}_\lambda k^b{}_{c\mu} - k^{ac}{}_\mu k^b{}_{c\lambda} \tag{3.13}$$

$$+\frac{1}{2}S^{ab}{}_{\mu\lambda}(p^{\mu\lambda}{}_{ab} - \pi^{\mu\lambda}{}_{ab}) + N^\lambda_{a\mu} q^\mu{}^a_\lambda \tag{3.14}$$

where $S^{ab}{}_{\mu\lambda}$ and $N^\lambda_{a\mu}$ are some local functions on X.

The corresponding Hamiltonian equations for a section $\sigma^\lambda_a(x), k^{ab}{}_\lambda(x), q^\mu{}^a_\lambda(x), p^{\mu\lambda}{}_{ab}(x))$ of the Legendre bundle Π^1 takes the form

$$\partial_\mu k^{ab}{}_\lambda = \frac{\partial\mathcal{H}}{\partial p^{\mu\lambda}{}_{ab}} = \frac{1}{2}(k^{ac}{}_\lambda k^b{}_{c\mu} - k^{ac}{}_\mu k^b{}_{c\lambda} + S^{ab}{}_{\mu\lambda}), \tag{3.15a}$$

$$\partial_\mu p^{\mu\lambda}{}_{ac} = -\frac{\partial\mathcal{H}}{\partial k^{ac}{}_\lambda} = k^b{}_{c\mu}(p^{\mu\lambda}{}_{ab} - p^{\lambda\mu}{}_{ab}), \tag{3.15b}$$

$$\partial_\mu \sigma^\lambda_a = \frac{\partial\mathcal{H}}{\partial q^\mu{}^a_\lambda} = N^\lambda_{a\mu}, \tag{3.15c}$$

$$\partial_\mu q^\mu{}^a_\lambda = -\frac{\partial\mathcal{H}}{\partial h^\lambda_a}. \tag{3.15d}$$

On the image of the Legendre morphism $\widehat{L}_{HE}$, we have

$$\partial_\mu p^{\mu\lambda}{}_{ac} = \partial_\mu \pi^{\mu\lambda}{}_{ac}, \tag{3.16a}$$

$$\partial_\mu q^{\mu a}_{\lambda} = 0. \tag{3.16b}$$

Equation (3.15a) leads to the relation

$$\frac{1}{2}(S^{ab}{}_{\mu\lambda} - S^{ab}{}_{\lambda\mu}) = \mathcal{F}^{ab}{}_{\mu\lambda}. \tag{3.17}$$

Substituting this relation into the equation (3.15d) and (3.16b), we obtain the Einstein equations. Substituting the equality (3.17) into the equations (3.15b) and (3.16a), we get the familiar relation between the connection $k^{ab}{}_\lambda$ and the tetrad field $\sigma^\mu_a(x)$. Equation (3.15c) shows that the tetrad field $\sigma(x)$ is parallel with respect to the connection N on the Higgs bundle Σ. It follows that N is a world connection which differs from the Lorentz connection induced by the connection $k(x)$ only in torsion.

Let us emphasize that the kinetic part of the multimomentum Hamiltonian form (3.14) of the classical gravity contains only the connection term

$$\frac{1}{2}p^{\mu\lambda}{}_{ab}(S^{ab}{}_{\mu\lambda} + k^{ac}{}_\lambda k^b{}_{c\mu} - k^{ac}{}_\mu k^b{}_{c\lambda}) + N^\lambda_{a\mu} q^{\mu a}_{\lambda}.$$

It follows that the Hamiltonian density of the classical gravity is equal to zero.

The symmetric part of equation (3.15a) represents the familiar gauge condition

$$\partial_\mu k^{ab}{}_\lambda + \partial_\lambda k^{ab}{}_\mu = \frac{1}{2}(S^{ab}{}_{\mu\lambda}(x) + S^{ab}{}_{\lambda\mu}(x)).$$

Thus, if we restrict ourselves to the pull-back atlases of the principal bundle P^L_s, we have the standard collection of gauge transformations.

In general, the fermion-gravitation complex possesses the following gauge transformations.

These are transformations of atlases Ψ^h_s (respect. Ψ^L) of the bundle P^L_s (respect. P^L) and transformations of holonomic atlases Ψ^T of the bundle LX. Given a tetrad field h, transformations of atlases Ψ^L_s and Ψ^L induce transformations of atlases Ψ^s of the bundle P^h_s and transformations of atlases Ψ^h of the bundles L^hX and LX.

Gauge bundle morphisms Φ_E of the bundle $\widetilde{E}$ are the associated general principal morphism of the bundle E^L which satisfy the following conditions.

(i) A morphism Φ_E preserves the fibration (3.1) and is projected to some general principal morphism Φ_Σ of the Higgs bundle Σ and to some diffeomorphism of the base X.

(ii) The morphism Φ_Σ is induced by the general principal isomorphism $(\Phi_X)_*$ of the linear frame bundle LX which is tangent to the diffeomorphism Φ_X.

For instance, $\Phi_E \in L_s(\Sigma)$ if Φ_X is the identity morphism.

Note that violation of the condition (ii) results in deviations of a gravitational field which we describe in the next Section.

3.2 Deviations of a Gravitational Field

If there are no other fields and a space-time decomposition is not considered, a tetrad gravitational field h as like as an ordinary Higgs field (Section 1.4) admits linear deviations in the second order in group parameters (since the Lorentz group L is a Cartan subgroup of the group GL_4). In Section 1.4, there were discussed deviations of a H-stable background field

$$(\psi^{h\Sigma}_\kappa)(x) = \sigma_0.$$

In contrast with the internal symmetry case, each gravitational field h can play the role of a Higgs background field because its Goldstone part fails to be removed by gauge transformations.

Let Ψ^T be a holonomic atlas of the principal bundle LX. Given a representer $h_\kappa(x) \in GL_4$ of the coset

$$(\psi^T_\kappa h)(x) = h_\kappa(x)\sigma_0 \in GL_4/L, \qquad x \in U_\kappa,$$

the deviations

$$\begin{aligned}(\psi^T_\kappa h')(x) = h'_\kappa(x)\sigma_0 &= \left[h_\kappa(x)\exp(\varepsilon(x))h^{-1}_\kappa(x)\right] h_\kappa(x)\sigma_0\\ &= h_\kappa(x)\exp(\varepsilon(x))\sigma_0\end{aligned}$$

of $h(x)$ are parameterized by elements $\varepsilon(x)$ of the Lie algebra of the group GL_4. Note that, if elements $\varepsilon(x)$ belong to the Lie algebra of the Lorentz group, one has

$$h'(x) = h(x),$$

but

$$h'_\kappa(x) \neq h_\kappa(x).$$

In the presence of the Dirac fermion matter, deviations of a tetrad gravitational field however fail to form a linear space even in the first order in group parameters. Given a gravitational field h, the representation γ_h (2.7) is written only with respect to atlases Ψ^h and the representers of gravitational field functions $h_\kappa(x)$ are tetrad functions (2.4). For different tetrad fields h and h', γ-matrices in the Dirac operator describe the nonisomorphic representations

$$\gamma = \gamma_h(h^a), \qquad \gamma^{\widetilde{a}} = \gamma_{h'}(h'^{\widetilde{a}})$$

of different cotangent vectors h^a and $h'^{\widetilde{a}}$ (Section 2.2). As a consequence, it is impossible to write

$$h'^\mu_{\widetilde{a}}\gamma^{\widetilde{a}} = (h^\mu_a + \varepsilon^\mu{}_a)\gamma^a$$

because one can not take γ-matrices beyond the brackets

$$(h'^\mu_{\widetilde{a}} - h^\mu_a)\gamma^{a=\widetilde{a}}.$$

In the Dirac operator, tetrad functions thereby do not admit linear deviations

$$h'^{\mu}_{\underset{\sim}{a}} = (Sh)^{\mu}_{\underset{\sim}{a}} = S^{a}{}_{\underset{\sim}{a}} h^{\mu}_{a} \neq (\delta^{a}_{\underset{\sim}{a}} + \varepsilon^{a}{}_{\underset{\sim}{a}}) h^{\mu}_{a}, \qquad S \in GL_4 \backslash L. \tag{3.18}$$

It follows that, in the presence of the Dirac fermion matter, tetrad gravitational *fields do not satisfy the superposition principle.*

The superposition principle is one of the corner-stones of quantum theory. On mathematical side, this principle requires fields or deviations of some background field form a vector space. This is the reason for our attention to linear deviations of a gravitational field.

Without regard to fermion fields, one can choose metric functions $g^{\mu\nu}$ as gravitational variables and can examine their small deviations (2.6). However, if a space-time decomposition is considered, these deviations also fail to form a linear space in general. Given a gravitational field g and a g-compatible space-time distribution $\mathbf{F}$, let k be a spatial part of the world metric g (Section 2.3). If a world metric g' results from a linear deviation

$$g' = g - \varepsilon$$

of g, one can require the spatial parts k' of g' to be a linear deviation

$$k' = k + \varepsilon_k$$

of k. It takes place if there exists a space-time distribution $\mathbf{F}$ compatible both with g and g'. In this case, we have

$$k' = k + \varepsilon + \frac{\omega \otimes \omega}{|g(\omega,\omega)|^2} \varepsilon(\omega,\omega),$$
$$\varepsilon(\omega,\omega) = \varepsilon^{\alpha\beta}\omega_\alpha\omega_\beta,$$

where ω is a generating form of the distribution $\mathbf{F}$. For instance, given a triple $(g, \mathbf{F}, g^R)$, every linear deviation

$$g'^R = g^R - \varepsilon^R$$

of the Riemannian metric g^R in this triple involves the linear deviation

$$g' = g + \varepsilon^R - 2\omega \otimes \omega \frac{\varepsilon^R(\omega,\omega)}{|g^R(\omega,\omega)|^2}$$

of the pseudo-Riemannian metric g in and its spatial part

$$k' = k - \varepsilon^R - \omega \otimes \omega \frac{\varepsilon^R(\omega,\omega)}{|g^R(\omega,\omega)|^2}$$

so that the triple $(g', \mathbf{F}, g'^R)$ is associated with the same distribution $\mathbf{F}$.

Obviously, there are pseudo-Riemannian metrics g and g' which fail to admit a common space-time distribution. Their superposition is accompanied by superposition of space-time distributions which we face, e.g., in the case of gravitational singularities of the caustic type (Chapter 2. Appendix).

At the same time, tetrad functions h^{μ}_{a} in the Dirac operator admit superposable deviations of the following type:

$$\begin{aligned} \widetilde{h}^{\mu}_{a} &= H^{b}{}_{a} h^{\mu}_{b} = (\delta^{b}_{a} + \sigma^{b}{}_{a}) h^{\mu}_{b} = H^{\mu}{}_{\nu} h^{\nu}_{a} \\ &= (\delta^{\mu}_{\nu} + \sigma^{\mu}{}_{\nu}) h^{\nu}_{a} = h^{\mu}_{a} + \sigma^{\mu}{}_{a}, \end{aligned} \tag{3.19}$$

$$\widetilde{L}_D = \widetilde{h}^{\mu}_{a}\gamma^{a} D_{\mu}. \tag{3.20}$$

These however are not the geometric deviations in the sense that the quantities $\widetilde{h}^{\mu}_{a}$ fail to be tetrad functions because, in comparison with expression (3.18), the indices a and b of $H^{b}{}_{a}$ correspond to the same reference frame Ψ^h. In contrast with tetrad functions, we have

$$\begin{aligned} &\widetilde{h}^{a}_{\mu} = g_{\mu\nu}\eta^{ab}\widetilde{h}^{\nu}_{b} = H_{\mu}{}^{\alpha} h^{a}_{\alpha}, \\ &\widetilde{h}^{\mu}_{a}\widetilde{h}^{a}_{\nu} \neq \delta^{\mu}_{\nu}, \qquad \widetilde{h}^{\mu}_{a}\widetilde{h}^{b}_{\mu} \neq \delta^{b}_{a}, \\ &\widetilde{g}^{\mu\nu} = \widetilde{h}^{\mu}_{a}\widetilde{h}^{\nu}_{b}\eta^{ab} = H^{\mu}{}_{\alpha}H^{\nu}{}_{\beta}g^{\alpha\beta}, \\ &\widetilde{g}_{\mu\nu} = \widetilde{h}^{a}_{\mu}\widetilde{h}^{b}_{\nu}\eta_{ab} = H_{\mu}{}^{\alpha}H_{\nu}{}^{\beta}g_{\alpha\beta}, \\ &\widetilde{g}^{\mu\nu}\widetilde{g}_{\mu\alpha} \neq \delta^{\nu}_{\alpha}. \end{aligned} \tag{3.21}$$

The quantity $\widetilde{g}$ in expression (3.21) is not a world metric. In comparison with the relation (2.6), for small deviations

$$\sigma^{\mu\nu} = \sigma^{a}{}_{b} h^{\mu}_{a} h^{b\nu},$$

we have

$$\widetilde{g}^{\mu\nu} \approx g^{\mu\nu} + \sigma^{\mu\nu},$$
$$\widetilde{g}_{\mu\nu} \approx g_{\mu\nu} + g_{\mu\alpha}g_{\nu\beta}\sigma^{\alpha\beta}.$$

In fibre bundle terms, we can describe the deviations (3.19) as follows. Let us consider associated general principal morphisms of the bundle E^L which are projected to arbitrary general principal isomorphisms Φ_Σ of the Higgs bundle Σ, but to the identity morphism of a world manifold X. Let these morphisms be accompanied by arbitrary bundle morphisms Φ_T of the cotangent bundle T^*X. If

$$\Phi_\Sigma \neq \mathrm{id}\,\Sigma,$$

these morphisms result in changes of a gravitational field. If

$$\Phi_\Sigma = \mathrm{id}\,\Sigma,$$

but

$$\Phi_\Sigma \neq \mathrm{id}\, T^*X,$$

we gain the deviations (3.19). To compare these two cases, let us regard morphisms Φ_Σ and Φ_T induced by the same principal isomorphism Φ of the principal bundle LX.

Since the bundle LX is assumed to be trivial, let us fix some global section $p(x)$ of LX. Then, every principal isomorphism (0.6) of LX can be written in the form:

$$\Phi\colon\ P_x = p(x)G^{-1} \to p(x)H_xG^{-1}, \qquad H_x = f_s(p(x)) \in G = GL_4.$$

This isomorphism induces the following associated principal morphism of the cotangent bundle T^*X:

$$\begin{aligned}(p(x)G^{-1} \times Gt)/G &\to (p(x)H_xG^{-1} \times Gt)/G \\ &= (p(x)G^{-1} \times GH_xt)/G, \qquad t \in T^*. \end{aligned} \tag{3.22}$$

Let us assume that

$$p(x) \in P^h_x, \qquad H_x \in G\backslash L, \qquad p(x)H_x \in P^{h'}_x$$

for some tetrad fields h and h'. If T^*X is regarded as the L-bundle M^hX, the morphism (3.22) can be written in the following two forms:

$$\begin{aligned}&\Phi_1\colon\ M^h_xX \ni (p(x)L^{-1} \times Lt)/L \to (p(x)L^{-1} \times LH_xt)/L \in M^h_xX, \\ &\Phi_2\colon\ M^h_xX \ni (p(x)L^{-1} \times Lt)/L \to (p(x)H_xL^{-1} \times Lt)/L \in M^{h'}_xX. \end{aligned} \tag{3.23}$$

Given a gravitational field h and the corresponding representation γ_h, the morphism Φ_1 induces the representation morphism

$$\widetilde{\gamma}_h = \gamma_h \circ \Phi_1.$$

With respect to the atlas

$$\{z^s(x), rz^s(x) = p(x)\}$$

of the principal spinor bundle P^h_s and the atlas

$$\{z^h(x) = p(x)\}$$

of LX, the morphism $\widetilde{\gamma}_h$ reads

$$\begin{aligned}\widetilde{\gamma}_h\colon\ \tau_a h^a(x) \otimes v(x) &= [z^s(x)L_s^{-1} \times (LH^a{}_b(x)\tau_a t^b \otimes (L_s v)]/L \to \\ &[z^s(x)L_s^{-1} \times L_s\gamma(H^a{}_b(x)\tau_a t^b \otimes v]/L = H^a{}_b(x)\tau_a\gamma^b v(x).\end{aligned}$$

Given an atlas Ψ^h of the principal bundle P^h, let us provide the principal bundle P^L with the pull-back Ψ^{hL} of the atlas Ψ^h. Let $\Psi^{Lh'}$ be the atlas of the

principal bundle $P^{h'}$ induced by the pull-back atlas Ψ^{hL}. We can compare the representations $\widetilde{\gamma}_h$ and $\widetilde{\gamma}_{h'}$ resulting from the representations γ_h by means of the associated morphisms (3.23):

$$\begin{aligned}&\gamma_h(h^a) = \gamma^a, \\ &\widetilde{\gamma}_h(h^a) = H^a{}_b\gamma^b, \\ &\gamma_{h'}(h^a) = \gamma_{h'}((H^{-1})^a{}_{\widetilde{b}}h'^{\widetilde{b}}) = (H^{-1})^a{}_{\widetilde{b}}\gamma^{\widetilde{b}}.\end{aligned}$$

The representation $\widetilde{\gamma}_h$ can be treated as deformation of the representation γ_h in a sense. The morphisms $\widetilde{\gamma}_h$ and γ_h define the γ-matrix representations of cotangent vectors on the same spinor fields ϕ_h. Therefore, deviations

$$H^a{}_b = \delta^a_b + \sigma^a{}_b$$

and their superposition

$$\sigma + \sigma'$$

can be defined.

The Dirac operator corresponding to the representation $\widetilde{\gamma}_h$ takes the form (3.20):

$$\begin{aligned}\widetilde{L}_D = \widetilde{\gamma}_h(dx^\mu)D_\mu\phi_h &= h^\mu_a(x)\widetilde{\gamma}_h(h^a(x))D_\mu\phi_h = h^\mu_a(x)H^a{}_b(x)\gamma^bD_\mu\phi_h \\ &= h^\mu_a(x)\gamma^aH^\nu{}_\mu(x)D_\nu\phi.\end{aligned}$$

Given a holonomic atlas, the functions $H^\nu{}_\mu(x)$ in this expression do not depend on a gravitational field, that is, tetrad functions h^μ_a and deviations $\sigma^\mu{}_\nu$ are independent dynamic variables.

Note that, if $H_x \in L \subset GL_4$, the representations $\widetilde{\gamma}_h$ and γ_h are isomorphic. For an infinitesimal element $H_x \in L$, we then have

$$\sigma_{ab} = -\sigma_{ba}.$$

The morphisms of cotangent bundles which do not change a world metric, the deviations (3.19) and the Dirac operator (3.20) appear in the gauge theory of the translation group (Section 4.2). We therefore may apply the Lagrangians of this theory in order to describe fields σ. Let us note that, to construct a Lagrangian of deviations ε of a gravitational field g, one usually use a familiar geometric Lagrangian of a metric field

$$g' = g - \varepsilon$$

where g is treated as a background field. In the case of deviations (3.19), we can not follow this method because quantities $\widetilde{g}$ (3.21) fail to be a true metric field.

We thus may say that the morphisms Φ_1 deform fibres of the cotangent bundle T^*X and thereby violate the identification

$$T^*X = M^hX$$

of the cotangent bundle with the Minkowski space bundle M^hX associated with the spinor bundle P^h_x.

Let us remark that the morphisms Φ_1 and Φ_2 are the equivalent transformations of the cotangent bundle regarded as the GL_4-bundle. Therefore, if world symmetries are not broken (e.g., there are no fermion fields), the bundle T^*X "loses" the structure of a Minkowski space bundle and the transmutations

$$\begin{aligned} M^h_xX &= (p(x)L^{-1} \times LH_xM)/L \to (p(x)G^{-1} \times GH_xT^*)/G \\ &= (p(x)H_xG^{-1} \times GT^*)/G \to (p(x)H_xL^{-1} \times LM)/L = M^{h'}_xX \end{aligned}$$

of deviations σ of a gravitational field h into a new gravitational field h' may take place. Given an atlas

$$\left\{z^{h'}(x) = z^h(x)H_x\right\},$$

we then have

$$h'^{\widetilde{a}}_\mu = H^a{}_b h^b_\mu = \widetilde{h}^a_\mu, \qquad h'^\mu_{\widetilde{a}} \neq \widetilde{h}^\mu_a, \tag{3.24}$$

where $h'^{\widetilde{a}}_\mu$ are tetrad functions of h' with respect to the atlas $\Psi^{h'}$ and h^a_μ are that of h with respect to Ψ^h.

Chapter 4

GAUGE THEORY OF THE TRANSLATION GROUP

For the first time, gauge theory of the Poincaré group was brought into play by T.W.B.Kibble, B.Frolov, D.Sciama at the beginning of 60s in order to generalize R.Utiyama's gauge version of gravity which had left open the question on the gauge status of tetrad gravitational fields. The Poincaré gauge approach (dominated gauge gravitation researches in the 60-80s [HEH, CHO, BAS, KAW, MIE]) however was no succeeded in identification of a gravitational field with a translation gauge potential [IVA 1983]. At the same time, gauge potentials of spatial translations appeared to acquire satisfactory physical utilization to the gauge theory of dislocations in continuous media [KAD]. By analogy, gauge potentials of the Poincaré space-time translations have been suggested to describe new fundamental interaction whose geometrical model is deformation of a world manifold [EDE, IVA 1987]. This interaction may contribute to standard gravitation effects, e.g., may result in the Yukawa type deviation of Newton's gravitational potential [SAR 1990].

By T^4, we further denote the additive group of the vector space $\mathbb{R}^4$. This is the group of translations in the vector space $\mathbb{R}^4$ provided with the canonical structure of an affine space. One identifies this affine space with the group space of T^4.

4.1 Gauge Models of the Poincaré Group

The translation group T^4 is the subgroup of the Poincaré group and the affine group $A(4,\mathbb{R})$. The Poincaré group is the group of isomorphisms of the affine Minkowski space. In Special Relativity, it plays the role of the fundamental dynamic group whose unitary representations describe the free particle states. This was the fact that motivated attempts to complete gauge theory of internal symmetries and Utiyama's gauge model of intrinsic spin symmetries with the gauge one of the Poincaré group. However, these attempts faced the feature of this group as the dynamic group. In contrast with internal and spin transformations altering field functions at a point, generators of the dynamic Poincaré symmetries are represented by the differential operators

$$T_\mu = \partial_\mu, \qquad L^{\rm orb}_{\mu\nu} = g_{\mu\alpha}x^\alpha\partial_\nu - g_{\nu\alpha}x^\alpha\partial_\mu. \tag{4.1}$$

They may be thought, on the one hand, as the generators of coordinate transformations and, on the other hand, as the generators of diffeomorphisms acting on field functions.

Authors of the first Poincaré gauge works adhered to the coordinate treatment of the Poincaré group generators (4.1). They combined Lorentz spin transformations with coordinate translations

$$x^\mu \to x^\mu + a^\mu.$$

Localization of these translations

$$x^\mu \to x^\mu + a^\mu(x)$$

reproduced the group of general coordinate transformations which induced, in turn, the holonomic subgroup of the atlas transformation group $GL_4(X)_A$.

The gauge model of Poincaré transformations (4.1) treated as diffeomorphisms was proposed in ref. [HEH]. Besides localization of group parameters, it modified the generators of the Poincaré group by replacing partial derivatives in expression (4.1) by the covariant ones

$$\partial_\mu \to D_\mu = \partial_\mu - \Gamma_\mu$$

where Γ was a Lorentz connection. Hence, localization of Poincaré transformations

$$\exp[t^\mu \partial_\mu + l^{\mu\nu}(L^{\rm orb}_{\mu\nu} + L^{\rm sp}_{\mu\nu})]$$

took nonconventional form

$$\exp[t^\mu(x) D_\mu + l^{\mu\nu}(x)(L'^{\rm orb}_{\mu\nu} + L^{\rm sp}_{\mu\nu})] \tag{4.2}$$

where $L'^{\rm orb}$ resulted from $L^{\rm orb}$ by replacement

$$\partial_\mu \to D_\mu.$$

This replacement seemed quite natural as generalization of translations in a flat space, but it violated the familiar commutation relations of the Poincaré group. For instance, the translation generators appeared to be noncommutative:

$$[D_\mu, D_\nu] \neq 0.$$

Obviously, the transformations (4.2) did not form the conventional gauge Poincaré group.

The conventional gauge technique can be applied to the Poincaré group if one ignores its physical role as a dynamic group and looks at it as an abstract structure group of a bundle.

Let AX be the affine frame bundle over a world manifold X. It is the principal bundle with the affine structure group $A(4,\mathbb{R})$. This bundle is associated with the affine tangent bundle A^TX. Moreover, we have

$$A^TX = AX/GL_4.$$

The affine bundle A^TX is isomorphic to the tangent bundle TX provided with the canonical structure of an affine bundle. Therefore, the structure group of AX and A^TX is reducible to the linear group GL_4.

Since the quotient space $A(4,\mathbb{R})/GL_4$ is homeomorphic to $\mathbb{R}^4$, in virtue of the well-known theorem [KOB], the structure group of any $A(4,\mathbb{R})$-principal bundle over a paracompact base is reducible to the linear subgroup GL_4. There is $1:1$ correspondence between reduced GL_4-subbundles of the affine frame bundle AX and global sections of the affine tangent bundle A^TX. In particular, the canonical reduction

$$\alpha:\ LX \to AX \tag{4.3}$$

corresponds to the global zero section $o(x)$ of A^TX. On the other hand, the homomorphism

$$A(4,\mathbb{R}) \to GL_4$$

entails the bundle morphism

$$\beta:\ AX \to LX$$

such that the composite $\beta\circ\alpha$ is the identity morphism of LX. We further restrict our consideration to bundle atlases $\{z_\kappa(x)\}$ of AX associated with the canonical reduction (4.3), that is,

$$z_\kappa(x) \in \alpha(LX).$$

The associated affine coordinates on the affine tangent bundle A^TX coincide with linear coordinates on TX, e.g.,

$$(x^\mu, u^\alpha = \dot{x}^\alpha).$$

Therefore, we further identify the affine tangent bundle A^TX with the tangent bundle TX provided with the canonical structure of an affine bundle.

Every affine connection A on TX is an associated principal affine connection. In virtue of the relation (0.48), it is represented by the sum

$$A = \Gamma + \sigma \tag{4.4}$$

of an associated principal linear connection Γ and a basic soldering form

$$\sigma:\ X \to T^*X \otimes TX,$$
$$\sigma = \sigma^\varepsilon{}_\mu(x)\partial_\varepsilon \otimes dx^\mu = \partial_\varepsilon \otimes \sigma^\varepsilon. \tag{4.5}$$

Coefficients $\sigma^\varepsilon{}_\mu$ of the form (4.5) are the coefficients of the local connection form of the affine part of the connection (4.4) with respect to the atlases associated with canonical reduction (4.3). We call A a canonical affine connection if $\sigma = \theta_X$.

Remark. Let $\overline{A}$ be a connection form of a principal affine connection on AX. Then, $\alpha^*\overline{A}$ is a 1-form on the total space P of LX which takes its values in the Lie algebra of the affine group $A(4,\mathbb{R})$. *The form* $\alpha^*\overline{A}$ *splits up into the sum*

$$\alpha^*\overline{A} = \overline{\Gamma} + B$$

of a linear connection form $\overline{\Gamma}$ and a $\mathbb{R}$-valued GL_4-equivariant horizontal 1-form $B(p)$ on P, that is,

$$\tau'^P \rfloor B(pg) = \tau^P \rfloor g^{-1}B(p), \quad p \in P, \quad g \in GL_4,$$

$$(\pi_{PX})_*\tau'^P = (\pi_{PX})_*\tau^P, \quad \tau \in T_pP, \quad \tau'^P \in T_{pg}P.$$

A soldering form σ in the relation (4.4) then is given by the expression

$$\tau \rfloor \sigma(x) = \tau^P \rfloor [p]_T B(p),$$
$$\tau = (\pi_{PX})_*\tau^P, \qquad x = \pi_{PX}(p).$$

Many authors tried to identify the soldering forms σ^ε (4.5) treated as translation gauge potentials with the tetrad forms (2.8) of some gravitational field h. Given h and the tangent bundle TX provided with the structure of the Minkowski space bundle M^hX, each basic soldering form σ on TX can be regarded as a basic soldering form

$$\sigma = \sigma^a_\mu(x) t^h_a \otimes dx^\mu = t^h_a(x) \otimes \sigma^a$$

on M^hX. For example, we have

$$\theta_X = t^a_h(x) \otimes h^a$$

for each gravitational field h. The soldering form θ_X itself however fails to fix a reduced L-subbundle L^hX of the principal bundle LX and therefore can not describe a gravitational field.

In the conventional gauge theory of the affine group, one thus faces the problem of physical interpretation both of a gauge translation potentials and of sections $u(x)$ of the affine tangent bundle TX. In field theory, no fields possess the transformation law

$$u(x) \to u(x) + a$$

under the translation subgroup of the Poincaré group.

At the same time, one observes such fields in the gauge theory of dislocations [KAD] which is based on the fact that, in the presence of dislocations, displacement vectors

$$u^m, \quad m = 1,2,3,$$

of small deformations are determined only with accuracy to gauge translations

$$u^m \to u^m + a^m(x).$$

In this theory, gauge translation potentials $\sigma^m{}_i$ describe plastic distortion, the covariant derivative

$$D_iu^m = \partial_iu^m - \sigma^m{}_i$$

coincides with elastic distortion, and the strength

$$\mathcal{F}^m{}_{ij} = \partial_i\sigma^m{}_j - \partial_j\sigma^m{}_i$$

describes dislocation density.

Equations of the gauge theory of dislocations

$$\partial^i(\mu D_iu^k + \frac{\lambda}{2}\delta^k_i D_ju^j) = 0, \tag{4.6}$$

$$\partial^i\mathcal{F}^k{}_{ij} = -\frac{1}{\varepsilon}(\mu D_ju^k + \frac{\lambda\delta^k_j}{2}D_mu^m) \tag{4.7}$$

can be derived from the gauge invariant Lagrangian density

$$\mathcal{L} = \mu D_iu^kD^iu_k + \frac{\lambda}{2}D_iu^iD_mu^m - \varepsilon\mathcal{F}^k{}_{ij}\mathcal{F}_k{}^{ij} \tag{4.8}$$

where μ and λ are the Lame coefficients of isotropic media. Equation (4.6) is the equilibrium equation, and equation (4.7) describes dislocation density. These equations fail to be independent of each other. Equation (4.6) is the divergence of equation (4.7) what reflects the fact that a displacement field $u^m(x)$ can be removed by gauge translations and, thereby, fails to be a dynamic variable.

In the spirit of the gauge dislocation theory, it was suggested that gauge potentials of the Poincaré translations may describe new geometric structure (sui generis deformations) of a world manifold.

4.2 Deformed Manifolds

Let the tangent bundle TX be provided with some affine connection (4.4). We consider the following two morphisms:

(i) *the morphism*

$$\widehat{o}\colon TX \to HTX \subset TTX$$

which is the morphism (0.40) restricted to the global zero section $o(x)$ of TX, that is,

$$\widehat{o} = A \circ o\colon o(X) \underset{X}{\times} TX \to HTX,$$

(ii) the geodesic morphism of TX onto X:

$$\zeta\colon TX \ni u \to \zeta(x,u,1) \in X, \qquad x = \pi_X(u),$$

where $\zeta(x,u,s)$ is the geodesic defined by the linear part Γ of the affine connection (4.4) through the point x in the direction u.

By deformation morphism of a world manifold X, we call the following tangent bundle morphism [SAR 1990]:

$$\rho = \zeta_* \circ \widehat{\sigma}\colon TX \to TX.$$

In the affine bundle coordinates (x^μ, u^α) on TX, this morphism is given by the expression

$$\begin{aligned}
&\rho_X\colon x \to x,\\
&\rho\colon \tau^\mu \frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x^\mu} + (\Gamma^\alpha{}_{\beta\mu} u^\beta + \sigma^\alpha{}_\mu)\frac{\partial}{\partial u^\alpha}\\
&\qquad \to \tau^\mu(\delta^\alpha_\mu + \sigma^\alpha{}_\mu)\frac{\partial}{\partial x^\alpha} = \tau^\mu H^\alpha{}_\mu \frac{\partial}{\partial x^\alpha}.
\end{aligned} \tag{4.9}$$

Here, we use the relations

$$\zeta^\mu(x, \lambda u, 1) = \zeta^\mu(x,u,\lambda), \qquad \lambda \in \mathbf{R},$$
$$\frac{\partial}{\partial u^\alpha}\zeta^\mu(x,u,1)|_{u=0} = \delta^\mu_\alpha.$$

and the expression

$$D_\mu u^\alpha|_{u=0} = (\partial_\mu u^\alpha + \Gamma^\alpha{}_{\beta\mu}u^\beta + \sigma^\alpha{}_\mu)|_{u=0} = \sigma^\alpha{}_\mu$$

for the covariant derivatives of a displacement field u.

Let E be a bundle over X and J^1E be the jet manifold of E. The deformation morphism (4.9) has the jet prolongation

$$j^1\rho\colon (x^\lambda, y^i, y^i_\lambda) \to (x^\lambda, y^i, H^\alpha{}_\lambda(x) y^i_\alpha)$$

projected to the identity morphisms of X and E. For instance, $j^1\rho(J^1E)$ is the affine bundle modeled on the vector bundle

$$(\widetilde{\rho}T^*X)\otimes VE$$

where

$$\widetilde{\rho}\colon dx^\lambda \to H^\lambda{}_\alpha(x) dx^\alpha$$

is the cotangent bundle morphism dual to ρ.

To define fields on a deformed manifold, we therefore can replace sections $w(x)$ and $w(y)$ of the bundles E^1 (0.28) and E^{01} (0.29) by sections

$$\widetilde{w}(x) = (j^1\rho \circ w)(x), \qquad \widetilde{w}(y) = (j^1\rho \circ w)(y).$$

If e is a section of the bundle E, we have

$$\widetilde{j^1e} = j^1\rho \circ j^1 e,$$
$$(\widetilde{j^1e})^i_\lambda = (e_* \circ \rho)^i_\lambda = H^\alpha{}_\lambda(x)\partial_\alpha e^i(x).$$

Let Γ be a connection on E and D be the covariant differential (0.44). On a deformed manifold, we get

$$\begin{aligned}
\widetilde{\Gamma}^i_\lambda(y) &= H^\alpha{}_\lambda \Gamma^i_\alpha(y)\\
\widetilde{D} &= dx^\lambda \otimes \widetilde{D}_\lambda = dx^\lambda \otimes H^\alpha{}_\lambda(x)(\partial_\alpha - \Gamma^i_\alpha(y)\partial_i)\\
&= H^\alpha{}_\lambda(x) \otimes dx^\lambda D_\alpha = (\widetilde{\rho}dx^\alpha)\otimes D_\alpha.
\end{aligned}$$

For instance, the Dirac operator on a deformed manifold takes the form

$$\begin{aligned}
\widetilde{L}_D &= \gamma_h \circ \widetilde{D} = \gamma_h(dx^\lambda)\otimes \widetilde{D}_\lambda = H^\alpha{}_\lambda(x)\gamma_h(dx^\lambda)\otimes D_\alpha\\
&= H^\alpha{}_\lambda(x) h^\lambda_a(x)\gamma^a D_\alpha.
\end{aligned}$$

This operator looks like the Dirac operator (3.20) in the presence of deviations (3.19) of a tetrad gravitational field if the morphism (3.22) coincides with the morphism $\widetilde{\rho}$ dual to the morphism (4.9). We therefore can apply Lagrangians of the field theory on deformed manifolds to deviations (3.19).

A Lagrangian density of a scalar field ϕ on the deformed manifold reads

$$\mathcal{L}_{(m)} = \frac{1}{2}(g^{\mu\nu}H^\alpha{}_\mu H^\beta{}_\nu D_\alpha\phi D_\beta\phi - m^2\phi^2)\sqrt{-g}.$$

Lagrangian densities $\mathcal{L}_{(g)}$ of the gravity and $\mathcal{L}_{(A)}$ of gauge potentials are constructed by means of the modified curvature

$$\widetilde{R}^{ab}_{\mu\nu} = H^\epsilon{}_\mu H^\beta{}_\nu R^{ab}{}_{\epsilon\beta}$$

and the modified strength

$$\widetilde{\mathcal{F}}^m_{\mu\nu} = H^\alpha{}_\mu H^\beta{}_\nu \mathcal{F}^m_{\alpha\beta}.$$

Remark. Given a principal connection A on a principal bundle P, its modified strength on the deformed manifold is defined to be

$$\widetilde{\mathcal{F}} = \widetilde{\rho}\circ\mathcal{F}\circ j^1 A^C.$$

The action functional and equations of motion of a point mass m_0 on the deformed manifold are given by expressions

$$S = -m_0 \int (g_{\alpha\beta} H^\alpha{}_\mu H^\beta{}_\nu v^\mu v^\nu)^{1/2} ds,$$
$$\frac{dv^\mu}{ds} + \widetilde{\Gamma}^\mu{}_{\alpha\beta} v^\alpha v^\beta = 0 \tag{4.10}$$

where v^μ is the 4-velocity and the quantities $\widetilde{\Gamma}$ look like the Christoffel symbols of the "metric"

$$\widetilde{g}_{\mu\nu} = H^\alpha{}_\mu H^\beta{}_\nu g_{\alpha\beta},$$

but the interval ds is defined by the true world metric g.

Let us note that, on the deformed manifold, a world metric and the volume form remain unchanged.

In the next Section, we discuss some field configurations on the deformed manifold. In the spirit of the gauge dislocation theory, we call translation gauge potentials the deformation fields.

4.3 Gauge Theory of the Fifth Force

A Lagrangian density $\mathcal{L}_{(\sigma)}$ of translation gauge potentials $\sigma^\varepsilon{}_\mu$ cannot be built in the Yang-Mills form because the Lie algebra of the affine group does not admit an invariant nondegenerate bilinear form. To construct $\mathcal{L}_{(\sigma)}$, one can apply the quantities $\sigma^\nu{}_\mu$ and $D_\alpha\sigma^\nu{}_\mu$ where D is the covariant differential defined by the linear connection Γ from expression (4.4). Since $\sigma^\nu{}_\mu$ are the coefficients of a local connection form, a linear connection Γ acts on the upper index of $\sigma^\nu{}_\mu$. Then, only the combination

$$\mathcal{F}^\alpha{}_{\nu\mu} = D_\nu\sigma^\alpha{}_\mu - D_\mu\sigma^\alpha{}_\nu$$

is possible. This is the torsion (0.51) of the connection Γ with respect to the soldering form σ.

The general form of a Lagrangian density $\mathcal{L}_{(\sigma)}$ of a deformation field is given by the expression

$$\mathcal{L}_{(\sigma)} = \frac{1}{2}[a_1 \mathcal{F}^\mu{}_{\nu\mu}\mathcal{F}_\alpha{}^{\nu\alpha} + a_2\mathcal{F}_{\mu\nu\sigma}\mathcal{F}^{\mu\nu\sigma} + a_3\mathcal{F}_{\mu\nu\sigma}\mathcal{F}^{\nu\mu\sigma}$$
$$+ a_4\varepsilon^{\mu\nu\sigma\gamma}\mathcal{F}^\varepsilon{}_{\mu\varepsilon}\mathcal{F}_{\gamma\nu\sigma} - \mu\sigma^\mu{}_\nu\sigma^\nu{}_\mu + \lambda\sigma^\mu{}_\mu\sigma^\nu{}_\nu]\sqrt{-g}.$$

where $\varepsilon^{\mu\nu\sigma\gamma}$ is the Levi-Civita tensor.

The mass-like term in $\mathcal{L}_{(\sigma)}$ is originated from the Lagrangian density (4.8) for displacement fields u under the gauge condition $u = 0$.

It seems natural to require the component $\mathbf{t}^{00}_{(\sigma)}$ of a metric energy-momentum tensor of deformation fields σ on the Minkowski space be positive. This requirement implies the following constraints on constants of $\mathcal{L}_{(\sigma)}$:

$$a_4 = 0, \quad a_1 \geq 0, \quad a_2 \geq 0, \quad a_3 + 2a_2 = 0, \quad \mu \geq 0, \quad \lambda \leq \frac{1}{4}\mu.$$

The Lagrangian density $\mathcal{L}_{(\sigma)}$ then takes the form

$$\mathcal{L}_{(\sigma)} = \frac{1}{2}[a_1\mathcal{F}^\mu{}_{\nu\mu}\mathcal{F}_\alpha{}^{\nu\alpha} + a_2\mathcal{F}_{\mu\nu\sigma}(\mathcal{F}^{\mu\nu\sigma} - 2\mathcal{F}^{\nu\mu\sigma}) - \mu\sigma^\mu{}_\nu\sigma^\nu{}_\mu + \lambda\sigma^\mu{}_\mu\sigma^\nu{}_\nu]\sqrt{-g}.$$

Remark. One can use the decomposition of the tensor $\mathcal{F}^\lambda{}_{\mu\nu}$ in three irreducible parts

$$\mathcal{F}^\lambda{}_{\mu\nu} = \widetilde{\mathcal{F}}^\lambda{}_{\mu\nu} + \frac{1}{3}(\delta^\lambda_\nu\mathcal{F}_\mu - \delta^\lambda_\mu\mathcal{F}_\nu) + \varepsilon^\lambda{}_{\mu\nu\alpha}\widetilde{\mathcal{F}}^\alpha,$$
$$\mathcal{F}_\mu = \mathcal{F}^\alpha{}_{\mu\alpha}, \qquad \widetilde{\mathcal{F}}^\alpha = \frac{1}{6}\varepsilon^{\alpha\mu\nu\sigma}\mathcal{F}_{\mu\nu\sigma},$$

where $\mathcal{F}_\mu$ is the spur, $\widetilde{\mathcal{F}}^\alpha$ is the pseudo-spur, and $\widetilde{\mathcal{F}}$ is the spur-free part of the tensor $\mathcal{F}$.

A total Lagrangian density includes Lagrangian densities $\mathcal{L}_{(m)}$ of mater fields, $\mathcal{L}_{(A)}$ of gauge potentials, and $\mathcal{L}_{(g)}$ of a gravitational field. Matter sources of a deformation field σ then are the following:

(i) a short canonical energy-momentum tensor of matter fields

$$-\frac{\delta\mathcal{L}_{(m)}}{\delta\sigma^{\mu\nu}} = -(H^{-1})_{\nu\beta}D_\mu\phi\frac{\partial\mathcal{L}_{(m)}}{\partial D_\beta\phi} = -(H^{-1})_{\nu\beta}(\mathbf{T}_{(m)}{}^\beta_\mu + \delta^\beta_\mu L_{(m)})$$

where $\mathbf{T}_{(m)}$ denotes a canonical energy-momentum tensor of matter fields;

(ii) a short metric energy-momentum tensor $\mathbf{t}_{(A)}$ of gauge potentials:

$$-\frac{1}{\varepsilon^2}a^G_{mn}\widetilde{g}^{\gamma\beta}H^\alpha{}_\mu\mathcal{F}^m_{\alpha\beta}\mathcal{F}^n_{\gamma\beta}\sqrt{-g}$$

where $\widetilde{g}^{\gamma\beta}$ is "metric" (3.21);

(iii) a curvature tensor

$$\kappa^{-1}g_{\nu\alpha}H^\varepsilon{}_\gamma R^{\alpha\gamma}_{\mu\varepsilon}\sqrt{-g}$$

of a gravitational field.

Let us restrict ourselves to the case of a small field σ. We neglect a gravitational field on the left-hand side of equations for σ and keep only σ-free terms in matter sources. Then, the Euler-Lagrange equations for a deformation field σ read

$$\begin{aligned}\frac{\delta\mathcal{L}_{(\sigma)}}{\delta\sigma^{\mu\nu}} &= a_1(\eta_{\mu\nu}\partial^\varepsilon\mathcal{F}^\alpha{}_{\alpha\varepsilon} - \partial_\mu\mathcal{F}^\alpha{}_{\alpha\nu}) + 2a_2\partial^\varepsilon(\mathcal{F}_{\mu\nu\varepsilon} - \mathcal{F}_{\nu\mu\varepsilon} + \mathcal{F}_{\varepsilon\mu\nu}) \\ &\quad - \mu\sigma_{\mu\nu} + \lambda\eta_{\mu\nu}\sigma^\alpha{}_\alpha = S_{\mu\nu}, \\ S_{\mu\nu} &= -(\mathrm{T}_{(m)\nu\mu} + g_{\nu\mu}\mathcal{L}_{(m)}) - \frac{1}{\varepsilon^2}a^G_{mn}g^{\beta\gamma}\mathcal{F}^m_{\mu\beta}\mathcal{F}^n_{\nu\gamma}\sqrt{-g} \\ &\quad + \kappa^{-1}R_{\mu\nu}\sqrt{-g}. \end{aligned} \tag{4.11}$$

One can replace the gravitation term in equation (4.11) by the right-hand side of the Einstein equations. Equations for σ then take the form

$$\frac{\delta\mathcal{L}_{(\sigma)}}{\delta\sigma^{\mu\nu}} = (\mathrm{t}_{(m)\nu\mu} - \mathrm{T}_{(m)\nu\mu}) - g_{\mu\nu}(\mathcal{L}_{(m)} + \frac{1}{2}\mathrm{t}_{(m)}) - g_{\mu\nu}\mathcal{L}_{(A)}.$$

In the case of scalar matter fields, we have

$$\mathrm{t}_{(m)\nu\mu} = \mathrm{T}_{(m)\nu\mu},$$
$$-g_{\mu\nu}(\mathcal{L}_{(m)} + \frac{1}{2}\mathrm{t}_{(m)}) = -\frac{1}{2}g_{\mu\nu}m^2\phi^2.$$

By analogy with equation (4.6), we can write the equilibrium equation

$$\partial^\nu\frac{\delta\mathcal{L}_{(\sigma)}}{\delta\sigma^{\mu\nu}} = -\mu\partial^\nu\sigma_{\mu\nu} + \lambda\partial_\mu\sigma^\alpha{}_\alpha = \partial^\nu S_{\mu\nu}. \tag{4.12}$$

Note that the right-hand side of equation (4.12) is equal neither to zero nor to a gradient quantity in general. At the same time, this is a pure gradient quantity if matter sources of the field σ are gauge potentials and scalar fields. These facts result in the important condition

$$\mu \neq 0, \qquad \mu \neq 4\lambda. \tag{4.13}$$

Since equations (4.11) are linear, their solutions differ from each other in solutions of the free field equations. In the case of a free field σ, equation (4.12) reads

$$-\mu\partial^\nu\sigma_{\mu\nu} + \lambda\partial_\mu\sigma^\alpha{}_\alpha = 0.$$

Taking into account this relation, one can bring equations (4.11) into the equations

$$4a_2\partial^\varepsilon(w_{\mu\varepsilon,\nu} + w_{\nu\mu,\varepsilon} - w_{\nu\varepsilon,\mu}) + 2a_1(w^\alpha{}_{\nu,\mu\alpha} - w^\alpha{}_{\mu,\nu\alpha}) - \mu w_{\mu\nu} = 0, \tag{4.14}$$

$$\begin{aligned} a_1\left[\frac{\lambda}{\mu} - 1\right][\eta_{\mu\nu}\Box e - e_{,\mu\nu}] + 2a_1(w^\alpha{}_{\nu,\mu\alpha} + w^\alpha{}_{\mu,\nu\alpha}) \\ - \mu e_{\mu\nu} + \lambda\eta_{\mu\nu}\sigma = 0 \end{aligned} \tag{4.15}$$

where

$$\Box = \partial^\alpha\partial_\alpha,$$
$$e_{\mu\nu} = \frac{1}{2}(\sigma_{\mu\nu} + \sigma_{\nu\mu}), \quad w_{\mu\nu} = \frac{1}{2}(\sigma_{\mu\nu} - \sigma_{\nu\mu}), \quad e = \sigma^\alpha{}_\alpha. \tag{4.16}$$

It seems natural to choose the solution $w = 0$ of equations (4.14). Equations (4.15) then can be written in the form

$$\begin{aligned} e_{\mu\nu} &= \frac{\mu - \lambda}{3\mu}(\eta_{\mu\nu}e - \frac{3a_1}{\mu}e_{,\mu\nu}), \\ \Box e + m^2 e &= 0, \qquad m^2 = \frac{\mu(\mu - 4\lambda)}{3a_1(\mu - \lambda)}, \end{aligned} \tag{4.17}$$

where the quantity m plays the role of a mass of deformation fields σ. In virtue of the condition (4.13), this mass is not equal to zero.

Equations (4.17) admit the following plane wave solutions

$$e_{\mu\nu} = \frac{\mu - \lambda}{3\mu}\left[\eta_{\mu\nu} + \frac{\mu - 4\lambda}{\mu - \lambda}\frac{p_\mu p_\nu}{p^2}\right]a(p)e^{ipx}, \qquad p^2 = m^2.$$

These solutions look promising in order to quantize both the deformation fields σ and the deviations (3.19) of a tetrad gravitational field.

Now, let us consider a model of a small deformation field σ and a small gravitational field

$$g = \eta + 2\varepsilon$$

if their matter source is a motionless point mass M. In this case, the right-hand side of equations (4.11) reads

$$-\frac{1}{2}\eta_{\mu\nu}\mathrm{T}_{(m)} = -\frac{1}{2}\eta_{\mu\nu}M\delta(r)$$

where, by (r, ϕ, θ), we denote spatial spherical coordinates.

Recalling the notations (4.16), we can rewrite equations (4.11) in the form

$$\frac{-a_1}{2}(e^\alpha{}_{\nu,\alpha\mu} - e^\alpha{}_{\mu,\alpha\nu}) + (4a_2 + \frac{a_1}{2})(w^\alpha{}_{\mu,\alpha\nu} - w^\alpha{}_{\nu,\alpha\mu}) - 4a_2\Box w_{\mu\nu} - \mu w_{\mu\nu} = 0,$$

$$\begin{aligned} a_1[\eta_{\mu\nu}(e^{\alpha\varepsilon}{}_{,\alpha\varepsilon} - \Box e) - \frac{1}{2}(e^\alpha{}_{\nu,\alpha\mu} + e^\alpha{}_{\mu,\alpha\nu} + w^\alpha{}_{\nu,\alpha\mu} + w^\alpha{}_{\mu,\alpha\nu}) + e_{,\mu\nu}] \\ - \mu e_{\mu\nu} + \eta_{\mu\nu}\lambda e = -\frac{1}{2}\eta_{\mu\nu}M\delta(r). \end{aligned}$$

These equations admit the static spherically symmetric solution with the following nonzero components

$$e_{rr} = -\frac{1}{\mu - \lambda}(3\lambda e_{00} + \frac{1}{2}M\delta(r)),$$

$$e_{\theta\theta} = -e_{00}r^2, \quad e_{\phi\phi} = -e_{00}r^2\sin^2\theta,$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}e_{00} - m^2 e_{00} = -\frac{1}{6}\frac{\mu}{a_1(\mu-\lambda)}M\delta(r),$$
$$e_{00} = \frac{\mu M}{24\pi a_1(\mu-\lambda)}\frac{e^{-mr}}{r}$$

where m is the mass (4.17).

Substituting this solution into equation (4.10), we obtain the modification of Newton's gravitational potential

$$\tilde{\varepsilon} = \varepsilon + e_{00} = -\frac{\kappa M}{8\pi r}\left(1 - \frac{\kappa^{-1}\mu}{3a_1(\mu-\lambda)}e^{-mr}\right).$$

Such a "Yukawa type" modification of Newton's gravitational potential (whose experimental verification received much attention in the 80s) is usually related to the hypothetical fifth fundamental force [FIS].

To contribute to standard gravitational effects, the fifth interaction must be as universal as gravity. Its matter source must contain a mass or other parts of the energy-momentum tensor. This interaction must be described by a massive classical field, though its mass is unusually small. A deformation field fits these conditions. For example, the mass (4.17) is expressed by means of constants of the Lagrangian density $\mathcal{L}_{(\sigma)}$ where μ and λ make the sense of coefficients of "elasticity" of a space-time.

As it was mentioned above, one can use Lagrangians and equations of this Section in order to describe the superposable deviation of a gravitational field.

For instance, let σ be small free deviations (3.19) described by the Lagrangian $\mathcal{L}_{(\sigma)}$ on the Minkowski space. Their energy-momentum tensor can be calculated. Transmutations of σ mentioned in Section 3.2 result in the gravitational field given by expression (3.24). We can describe the gravitational field g' as a small gravitational potential

$$2\varepsilon^{\mu\nu} = (g'^{\mu\nu} - \eta^{\mu\nu}) = \sigma^{(\mu\nu)}, \qquad 2\varepsilon_{\mu\nu} = (g'_{\mu\nu} - \eta_{\mu\nu}) = -\sigma_{(\mu\nu)}$$

on the background Minkowski space. The energy-momentum tensor of the fields ε is defined and differs from that of deviations σ. It follows that, transmutations of the deviations (3.19) into a new gravitational field may violate the usual energy conservation law.

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