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What is a mathematical structure?

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Abstract. We suggest a modified definition of a mathematical structure which is based on the notion of a relation on a set and which generalizes the definition of a relational system in set theory. Morphisms and functions are structures in this sense, and this fact provides a wide circle of applications of this notion of a structure to mathematical physics.

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Introduction

A notion of the mathematical structure was introduced at the beginning of XX century. However, for a long time, mathematical objects were believed to be given always together with some structure, not necessarily unique, but at least natural (canonical). And only a practice, e.g., of functional analysis has led to conclusion that a canonical structure need not exist. For instance, there are different “natural” topologies of a set of rational numbers, different smooth structures of a four-dimensional topological Euclidean space, different measures on a real line, and so on [1].

In mathematics, different types of structures are considered. These are an algebraic structure, a topological structure, cells whose notion generalizes the Boolean algebras and so on. In the first volume of their course, Bourbaki provide a description of a mathematical structure which enables them to define "espece de structure" and, thus, characterize and compare different structures [2]. However, this is a structure of mathematical theories formulated in terms of logic. We aim to suggest a wider definition of a structure which absorbs the Bourbaki one and the others, but can not characterize different types of structures (see Section 2). This definition is based on a notion of the relation on a set (see Section 1), and it generalizes the definition of a relational system in set theory [3].

Morphisms and functions are structures in this sense that provides a wide circle of perspective applications of this notion of the structure to mathematical physics [4].

In particular, let us mention the notions of the universal structure on a set (see Section 2) and the abstract structure on its own elements (see Section 3.3). One can show that any structure is a constituent of a universal structure, and that any structure admits an exact representation as a constituent of some abstract structure.

Though we follow the von Neumann – Bernays – Gödel set theory, structures on sets only are considered unless otherwise stated (see Section 4). This is sufficient in order to investigate real, e.g., physical systems.

1. Relations

Let X be a set. A *propositional function* R, taking their values 1 ("true") and 0 ("false") on an n-times direct product $X^n = \overset{n}{\times} X$ of this set is called the n-ary *relation* on X. Elements $x_1,...,x_n \in X$ are said to be in a relation R, i.e., $x_1...x_n R$, if R is true at an element $(x_1,...,x_n) \in X^n$, i.e., $R(x_1,...,x_n) = 1$. A subset of X^n where a relation R is true ($R = 1$) is said to be the *domain* of R. A relation is uniquely defined by its domain. Therefore, we denote a domain by the same symbol R as a relation, and call it the relation, too.

Let $K \subset X$ be a minimal subset of X such that any element of K is in a relation R to some elements of X, for instance, to itself. Let us call K the *content* of R and its elements the *objects* of this relation. In general, a content K does not coincide with a set X, named a *universe*.

Let us note that, in the framework of the von Neumann – Bernays – Gödel set theory, a *proper class* U of all sets also can be a universe of a relation because its direct product $U \times U$ is well defined. Herewith, there is no additional problem if a content of a relation is a set (see Section 4), but one can also consider a domain $R \subset U^n$ to be a proper class.

Let us list some important examples of relations.

(i) If a domain of a relation R is empty, i.e. $R = 0$ on X. It characterizes the absence of any n-ary relation between elements of X, and therefore it is called the *empty relation*.

The absence of a relation sometimes is convenient in order to characterize an empty relation. Hereafter, a relation is assumed to be non-empty unless otherwise stated.

(ii) A 1-ary relation $R \subset X$ is called the *selection relation*. A content of this relation coincides with its domain. In particular, a set $R = X$ itself can be a selection relation. A selection relation also is exemplified by a one point set $R = \{x\}$ which contains only an element $x \in X$ which thus is both a relation and its object.

A selection relation R defines a binary relation $R^2 = R \times R \subset X^2$ on X such that a condition xR implies xxR^2, whereas conditions xxR^2 and $x'x'R^2$ are equivalent to the ones $xx'R^2$ and $x'xR^2$. This relation R^2 is called the *similarity relation*, and its objects are said to be the *similar ones*.

(iii) A binary relation R on a set X is called the *equivalence relation* if: (a) xxR for all $x \in X$, i.e., X is a content; (b) if xyR and yzR, then xzR; (c) xyR is equivalent to yxR. A subset $F \subset X$ is called the *equivalence class* of R if its elements are in a relation R to each other and to no others. Equivalence classes form a decomposition of a content X. A set X/R of this equivalence classes is called the *factor set* or the *quotient* of X with respect to R.

For instance, the above mentioned similarity relation $R^2 \subset X^2$ is the equivalence one possessing a unique equivalence class R.

(iv) A binary relation R on X is called the *relation of a particular order* if it satisfies the following conditions: (a) xxR for all $x \in X$; (b) xyR and yzR result in xzR; (c) xyR and yxR leads to $x = y$. One usually writes $x \leq z$ if xzR. An element $x \in X$ is called *minimal* (resp. *maximal*) if a condition $z \leq x$ (resp. $x \leq z$) leads to an equality $z = x$.

A relation of a partial order is said to be the *relation of an order* (or *linear order*) if any two elements of X are in this relation.

(v) A binary relation R on a set X is called the *morphism relation* if xqR and xpR implies that the equality $q = p$. Such a relation defines a map $x \to q$ of a set X to itself. Conversely, any map $\varphi : X \to X$ yields a morphism relation R_φ such that $x\varphi(x)R_\varphi$.

(vi) A 3-ary relation R on X is said to be the *multiplication relation* if $xyqR$ and $xypR$ leads to $q = p$. This relation provides X with a multiplication $x \circ y$ such that $xy(x \circ y)R$. Conversely, any multiplication $\mu : X^2 \to X$ endows X with a 3-ary relation R_μ such that $xy\mu(x,y)R_\mu$ for all $x, y \in X$. For instance, if $yx\mu(x,y)R_\mu$ for all $x, y \in X$, then a multiplication $\mu : X^2 \to X$ is commutative.

(vii) In a more general setting, let us consider a map $\mu : X^n \to X$. Its image $\mu(x_1,...,x_n)$ is called the *composition of elements* $x_1,...,x_n \in X$. This map yields an $(n+1)$-ary *composition relation* R_μ such that $x_1...x_n\mu(x_1,...,x_n)R_\mu$ and that conditions $x_1...x_n qR_\mu$ and $x_1...x_n pR_\mu$ result in an equality $q = p$.

For instance, a morphism relation is the binary composition one, whereas a multiplication relation is a 3-ary composition relation.

Obviously, every n-ary relation on a set X is a selection relation on a set X^n, and *vice versa*. Given a pair of sets X and V, one therefore can define a *V-valued n-ary relation* on X as a selection relation on $X^n \times V$.

In particular, let us consider a selection relation R on $X^n \times V$ such that $x_1...x_n qR_\mu$ and $x_1...x_n pR_\mu$ imply $q = p$. By analogy with the case of $V = X$, let us call it the n-ary *V-valued composition relation* on X. For instance, an $(n+1)$-ary composition relation is an n-ary X-valued composition relation. Any map of sets $X \to V$ yields a composition relation on $X \times V$, and *vice versa*.

If a composition relation is not required to satisfy that $x_1...x_n qR_\mu$ and $x_1...x_n pR_\mu$ implies $q = p$, can think of it as being a *multivalued composition relation*, and any relation is so.

Note that, unless otherwise stated, we follow a standard notion of the uniquely valued map $X \to V$. If *multivalued maps* are considered, any selection relation φ on $X \times V$ defines both a map $X \to V$ and a map $V \to X$ because there is no difference between an image and a domain of a multivalued map φ. Therefore, we agree to call a multivalued map the *correspondence of sets*. Let us denote $\pi_X : X \times V \to X$ and $\pi_V : X \times V \to V$, and let us call $\varphi_X = \pi_X(\varphi) \subset X$ and $\varphi_V = \pi_V(\varphi) \subset V$ the *projections of a correspondence* to X and V, respectively.

Let a set X be provided with n-ary relations R and S. We think of their pair (R, S) as being a *system of relations* on X. Herewith, different n-plets of elements of X which are either in a relation R or a relation S can obey certain conditions, called the *constraints*. On another hand, if n-plets of elements of X are regarded both in a relation R and a relation S, we come to the following *combinations of relations*.

(A) A union $R \cup S$ of domains R and S defines a relation $R \cup S = R + S - RS$ on a set X called the *union of relations*. Elements of X are in this relation if they are either in a relation R or a relation S. Therefore a content of $R \cup S$ is a union of contents of R and S.

(B) An intersection $R \cap S$ of domains R and S defines a relation $R \cap S = RS$ on X called the *intersection of relations*. Elements of X are in this relation if they are both in a relation R and a relation S. Therefore a content of a relation $R \cap S$ is an intersection of contents of relations R and S. If this intersection is empty, we have an empty relation $R \cap S$.

(C) Let $R \le S$, i.e., $R \subset S$. Then a relation R is called the *particular of a relation* S. Its content belongs to a content of S. There exists an n-ary relation $S = X^n$ on X such that any n-ary relation on X is its particular. It is called the *universal n-ary relation*.

For example, a set of integer numbers **Z** with operations of a product $r \circ r'$ and a sum $r + r'$ is characterized by a pair of 3-ary relations (R, S) such that $rr'(r \circ r')R$ and $rr'(r + r')S$, which also obey the *distribution condition* $(r \circ r')(r \circ r'')(r \circ (r' + r''))S$ as a constraint. A union $R \cup S$ of these relations is a subset of elements $(r, r', r \circ r')$ and $(r, r', r + r')$ of $\boldsymbol{Z}^3$ which is no multiplication relation. Their intersection $R \cap S$ is a subset of $\boldsymbol{Z}^3$ which consists of elements (r, r', q) such that $q = r \circ r' = r + r'$, that is, it contains only two elements (0,0,0) and (2,2,4).

The notions of the system and the combination of relations on a universe X are straightforwardly extended to an arbitrary family of n-ary relations $(R_j, j \in J_n)$ characterized by some subset J_n of a set 2^{X^n} of all subsets of X^n.

Let us now define morphisms of relations. In mathematics, by morphisms are meant the maps of sets with structures. Here we consider a more general case of morphisms of relations under correspondences of sets.

Let $X \leftrightarrow Y$ be some correspondence of sets X and Y given by a selection relation $\varphi \subset X \times Y$ on their product. For the sake of simplicity, a symbol φ also stands for the according correspondence of their direct products $X^n \leftrightarrow Y^n$.

A correspondence $X \leftrightarrow Y$ is a *map* $X \to Y$ of a set X to a set Y if any element of X is correspondent to a unique element of Y. Due to this condition, one can define a composition $X \to Y \to Z$ of maps $X \to Y$ and $Y \to Z$ which is a map $X \to Z$. In comparison with maps, a composition of correspondences $X \leftrightarrow Y$ and $Y \leftrightarrow Z$ is not well defined because different elements y and y' of Y can correspond to an element $x \in X$ and it may happen that they, in turn, are correspondent to different subsets of Z. As a consequence, x is correspondent through y and y' to these different subsets.

Let R be an n-ary relation on a set X defined by a selection relation on a set X^n. Given the notation $\pi_X : X^n \times Y^n \to X^n$ and $\pi_Y : X^n \times Y^n \to Y^n$, let us extend R as $R \circ \pi_X$ on $X^n \times Y^n$, then let us restrict its domain $(R \circ \pi_X)_\varphi = (R \circ \pi_X) \cap \varphi$ to a correspondence $\varphi \subset X^n \times Y^n$ and, finally, project it as $\pi_Y((R \circ \pi_X)_\varphi)$ to Y^n. This is a domain of an n-ary relation on a set Y which is called the *image* of a relation R under a correspondence φ. A domain of this image R_φ obviously belongs to a projection φ_Y of a correspondence φ in Y^n.

In other words, an image R_φ on Y of a relation R on X consists only of elements of Y^n which correspond to some points of a domain of R in X^n.

In particular, let a correspondence φ be a map of sets $X \to Y$, and let R be a relation on X. Then a domain of its image R_φ on Y is an image $\varphi(R)$ of a domain of R under a map φ. Now, if S is a relation on Y, then a domain of its image S_φ on X is an inverse image $\varphi^{-1}(S)$ of a domain S.

Given a correspondence $\varphi : X \leftrightarrow Y$, if a set Y is provided with some n-ary relation S and R_φ is its particular, then φ is said to be the *morphism of a relation* R to a relation S. For instance, if Y is endowed with an image R_φ, a morphism of R to its image takes place. It should be emphasized that, since a composition of correspondences is ill defined, a composition of relation morphisms is so.

Let sets Y and X admit n-ary relations S and R, respectively. A correspondence $\varphi : X \leftrightarrow Y$ is called the *isomorphism of relations* S and R if an image R_φ of R coincides with S and an image S_φ of S coincides with R, i.e. $R_\varphi = S$ and $S_\varphi = R$. Herewith, a correspondence φ need not be a bijection of sets.

Isomorphic relations on the same set are called *equivalent* if their isomorphism is a bijection of their contents.

2. Structures

Turn now to a notion of the structure on a set.

Let X be a set and J_n some subset of a set 2^{X^n} of all subsets of X^n, $n = 1,\ldots$. In particular, J_n can be an empty set. Let R_{j_n} denote a subset of X^n which is an element $j_b \in J_n$. We think of it as being a domain of an n-ary relation on a universe X. Then these subsets constitute a system ($R_{j_n \in J_n}$) of n-ary relations on a set X. A family ($R_{j_1 \in J_1}, \ldots, R_{j_n \in J_n}, \ldots$) of such systems ($R_{j_n \in J_n}$), $n = 1,\ldots$, of relations is called the *structure* of *characteristic* ($J_1, \ldots, J_n, \ldots$) on a *universe* X. Since such a structure is uniquely defined by its characteristic ($J_1, \ldots, J_n, \ldots$), we usually say that this is a structure ($J_1, \ldots, J_n, \ldots$).

This notion of the structure generalizes that of the *relational system* on a set X when J_n are non-empty finite sets for a finite set of numbers n [3].

Let $K \subset X$ be a minimal subset of X such that any element of K is in some relation R_{j_n} to some elements of X, for instance, to itself. We say that K is a *content of a structure* and its elements are *objects of a structure*. Relations R_{j_n} (and, equivalently, elements $j_n \in J_n$) are called the *elements of a structure*. These elements form a set

$$J = (J_1, \ldots, J_n, \ldots) = \overset{n}{\times} J_n .$$

It may happen that they obey some constraints, given by a family of propositional functions with a help of logical symbols and quantifiers.

Let us list some important examples of a structure.

(i) A set X without structure can be characterized as a set with an *empty structure* when all J_n are empty sets.

(ii) If all elements of a structure are selection relations, i.e., J=(J_1), it is called a *selection structure.* For instance, any set X admits a selection structure $J=(J_1=\{X\})$ and a selection structure $J=(J_1=\bigcup_{x\in X}\{x\})$.

(iii) Let X be a topological space characterized by a system of open subsets {R_j, $j\in J_1\subset 2^X$}. Then a system of selection relations ($R_{j\in J}$) provides X with a structure (J_1) which is a *topological structure* on X.

(iv) Let R be an n-ary relation on a set X. It provides X with a structure (R)=(J_n={j_R}) where {j_R} is a subset of 2^{X^n} possessing only one element j_R corresponding to R. Given an element $x\in X$, a relation R yields an ($n-1$)-ary relation $R_x\subset X^{n-1}$ such that $x_1...x_{n-1}R_x \Leftrightarrow x_1...x_{n-1}xR$. Then a system of relations J_R=(R_x, $x\in X$) is a structure on a set X which is called a *structure modelled over a relation* R.

Since there is one-to-one correspondence $x\leftrightarrow R_x$ between the objects $x\in X$ of this structure and its elements R_x, one can think of J_R as being a structure on its own elements, i.e., an abstract structure (see below).

(v) Let R be a binary relation of a particular order on a set X. Then any element $x\in X$ defines a selection relation R_x which contains all elements $z\in X$ such that $z\leq x$. As a result, a system of relations (R_x, $x\in X$) constitute a *structure of a particular order* on a set X.

(vi) Any map of sets $\mu: X\to Y$ is a structure on a set $X\times Y$ given by a morphism relation R_μ. It is called the *morphism structure*. Accordingly, a structure modelled over a morphism relation R_μ is defined by a system of relations $R_y=\mu^{-1}(y)$, $y\in Y$, on X. Its content is X, and its elements form a set $\mu(X)\subset Y$. It is called the *structure of inverse images.*

(vi) In particular, if Y in item (v) is a field of real numbers ***R***, then R_μ defines a real function on X.

The fact that maps and functions are structures provide a wide circle of applications of this notion of a structure. In particular, *differential geometric structures* (sections of fibre bundles, connections and so on) are morphisms structures.

There are the following two examples of *algebraic structures* defined by composition relations.

(i) Let a set X be endowed with a 3-ary multiplication relation R_μ such that $ab\mu(a,b)R_\mu$ for all $a,b\in X$. They define a *multiplication structure* (R_μ) on X. Let a relation R_μ obey the following conditions: (a) if $abcR_\mu$, $cdgR_\mu$ and $bdqR_\mu$, then $aqgR_\mu$ for all $a,b,d\in X$; (b) there exists an element $e\in X$, called the unit, such that $eaaR_\mu$ and $aeaR_\mu$ for all $a\in X$; (c) for each element $a\in X$, there exists an element $a^{-1}\in X$, called the inverse of a, such that $aa^{-1}eR_{\mu}$ and $a^{-1}aeR_{\mu}$. Then a multiplication structure (R_{μ}) is a *group structure* on a set X. For instance, if $abcR_{\mu}$ implies $bacR_{\mu}$ for all elements $a,b \in X$, then a group is commutative.

(ii) Let a set X be provided with a 3-ary relation S, defined a structure of a commutative group $xx'(x+x')S$, and a 3-ary multiplication relation $xx'(x \circ x')R$. Let us assume that they satisfy a distribution condition $(x \circ x')(x \circ x'')(x \circ (x'+x''))S$. Then a structure ($R,S$) is a *structure of an algebra* on X. Such a structure is exemplified by the above mentioned algebra of integer numbers **Z**.

Given a structure ($J_1,\ldots,J_n,\ldots$) on a set X, let J_n' be subsets of J_n for all $n=1,\ldots$. Then X is provided with a structure ($J_1',\ldots,J_n',\ldots$) named the *constituent of a structure* ($J_1,\ldots,J_n,\ldots$). A constituent of a structure is said to be *proper* if it does not coincide with a structure itself. A structure is called *elementary* if it admits a proper constituent. A structure is elementary iff it is characterized by one element.

Let a set X be endowed with a structure J=($J_1,\ldots,J_n,\ldots$). Let us consider a map of J to a set J'=($J_1',\ldots,J_n',\ldots$) such that an image of any element j of a structure J in J' represents a subset $R_j' \subset R_j$ of a domain R_j of j Sets R_j' treated as relations constitute a structure J' on a set X called the *particular of a structure* J. Obviously, a constituent of a structure is its particular because its elements are either elements of J or empty relations.

For instance, let us consider a set of integer numbers **Z** endowed with an algebra structure (S,R). Its constituent is a structure of an additive group (S) on **Z**. Let R' be a subset of R which contains elements $rr'(r \circ r')R$ where r are positive numbers only. Then a structure (S,R') is a particular of (S,R).

Structures on a set X constitute a set that we denote S^X. A notion of the particular of a structure provides it with a relation of a particular order P called the *particularity relation*. Its minimal elements is an empty structure, and the maximal one is a structure J_{UX}=($2^X,\ldots,2^{X^n},\ldots$) called the *universal structure on a set* X. Elements of this universal structure are arbitrary relations on a set X, i.e. all subsets of sets X^n for all *n*. Therefore any structure on a set X is a constituent of the universal structure J_{UX}. Herewith, a set X can be identified with a subset

$$\overline{X} = \bigcup_{x \in X} \{x\} \subset 2^X$$

of a set 2^X, i.e., with a constituent $J=(J_1=\overline{X})$ of a universal structure J_{UX} on X itself.

A particularity relation as a relation of a particular order provides a set of structures S^X on a set X with a structure of a particular order which is called the *particularity structure*. If there is no danger of confusion, this term further stands for any structure of a particular order modelled over a particularity relation. Being a structure of a particular order, a particularity structure is an abstract structure (see below).

3. Compositions of structures

In a general setting, a *composition of structures* is defined as a composition of elements of a set S^X of structures. Let us consider some examples of such a composition.

3.1. Image of a structure

Following the case of relations, we consider images and morphisms (see below) of structures under correspondences of sets.

Let $\varphi: X \leftrightarrow Y$ be a correspondence of sets X and Y defined by a selection relation $\varphi \subset X \times Y$. It yields a unique correspondence of sets $2^{Y^n} \leftrightarrow 2^{X^n}$ denoted by the same symbol φ. Let a set X be provided with a structure $J =(J_1,\ldots,J_n,\ldots)$. Then images $(R_j)_\varphi$ of relations R_j, which are elements of a structure $(J_1,\ldots,J_n,\ldots)$ on a set X yield a structure $J_\varphi =((J_1)_\varphi,\ldots,(J_n)_\varphi,\ldots)$ on a set Y called the *image of a structure* J or the *induced structure* under a correspondence φ.

For instance, if $Y \subset X$ is a subset, then an induced structure on it is defined by intersections $R_j \cap Y^n$ of domains of n-ary relations R_j of a structure on X, i.e., by the restrictions of functions R_j to $Y \subset X$. It is called the *restriction of a structure* on X to $Y \subset X$.

If $X \subset Y$, then an image of a structure J onto Y is this structure itself regarded as a structure on a universe Y whose content is $X \subset Y$. It is called the *extension of a structure* J.

3.2. Equivalent structures

Let X be provided with two structures $(I_1,\ldots,I_n,\ldots)$ and $(J_1,\ldots,J_n,\ldots)$. These structures are said to be *equivalent* if there exists a map φ of a set X to itself such that it is a bijection of contents of these structures and $\varphi: I_n \leftrightarrow J_n$ is a bijection for all n, i.e., φ is an equivalence of relations constituting these structures. The *identity equivalence* when a map φ is an identity morphism $\varphi(I_n) = \mathrm{Id}(I_n)$ of all domains of relations is said to be a *symmetry of a structure*.

For instance, a selection relation R on a set X uniquely defines a binary similarity relation R^2 on X, and *vice versa*. However, these relations R and R^2 yield non-equivalent structures on X.

Let a set X admit a group structure (R) such that $ab(a \circ b)R$ for all $a,b \in X$. Any element $a \in X$ defines a binary morphism relation ${}_aR$ on X such that $b(a \circ b)_aR$ for all

$b \in X$. A set of relations ${}_aR$, $a \in X$, defines a structure $J = (J_2 = \bigcup_{a \in X} {}_aR)$ on X which is a *structure of a left regular representation* of a group X. It is not equivalent to a group structure (R) on X. At the same time, relations ${}_aR$, $a \in X$, constitute a set which is canonically bijective to a set X and, thus, is provided with a group structure, isomorphic to a structure (R) (see below), and a structure of a left regular representation. Similarly, a structure of a right regular representation is introduced. It is not equivalent to that of a left regular representation unless a group is commutative.

A structure is not equivalent to its proper constituent, but it may happen that it is equivalent to its particular.

One can show that, for equivalent structures, constraints between their elements are maintained [3].

We agree to say that equivalent structures on a set X belong to the same *type*. Since an equivalence of structures is an equivalence relation, the types of structures on a set X constitute a set of equivalence classes of structures on X with respect to their equivalence. For instance, a structure and its proper constituent belong to different types of structures.

3.3. Morphisms of structures

Let X and Y be provided with structures $J = (J_1, \ldots, J_n, \ldots)$ and $I = (I_1, \ldots, I_n, \ldots)$, respectively. One says that a correspondence of sets $\varphi : X \leftrightarrow Y$ defines a *morphism of structures* $\varphi : J \to I$ if an image J_φ on Y of a structure J on X is a particular of a structure I on Y. Herewith, an image I_φ on X of a structure I on Y need not be a particular of a structure J, i.e., a morfism φ of structures $J \to I$ fails to be a morfism of a structure I to J in general.

For instance, this definition of morphisms of structures reproduces standard definitions both of morphisms of algebraic structures and continuous maps of topological spaces.

Note that, if a set Y is provided with an image of a structure on X under a correspondence $\varphi : X \leftrightarrow Y$, we have a morphism of a structure on X to its image.

It should be emphasized that, since a composition of correspondences of sets is ill defined, a composition of morphisms of structures is so.

If $\varphi : X \leftrightarrow Y$ is a correspondence such that $(J_n)_\varphi = I_n$ and $(I_n)_\varphi = J_n$ for all n, then φ is said to be the *isomorphism of structures*. Obviously, equivalent structures are isomorphic.

A morphism of structures $\varphi : J \to I$ under a correspondence $\varphi : X \leftrightarrow Y$ is called a *representation of a structure* J on a universe Y if its image J_φ is a constituent of a structure I. In particular, a morphism of a structure to its image is a representation. If $J_n \to (J_n)_\varphi$ is a bijection for all n, then a representation is said to be *exact*.

If a set Y is provided with a universal structure J_{UY}, any correspondence $\varphi : X \leftrightarrow Y$ defines a morphism of structures, and an image J_{φ} on Y of any structure J on X is a constituent of this universal structure J_{UY} on Y, i.e., we have a representation of a structure J in J_{UY}.

In particular, let a set X is provided with a structure J =($J_1, \ldots, J_n, \ldots$). Let $\varphi : X \to \overline{X} \subset J_{UX}$ be its canonical injection onto a subset $\overline{X} = \bigcup\limits_{x \in X} \{x\}$ of a set of elements of a universal structure J_{UX} on X. Then it yields an isomorphic structure J_{φ} on $\overline{X}$ and, consequently, an exact representation J_{φ} of a structure J on elements of J_{UX}.

Thus, any structure on a set X admits an exact representation on elements of a universal structure on X, and a universal structure on X is carried out on its own elements.

A structure on its own elements is called *abstract*. It means that relations constituting an abstract structure are propositional functions on a set of these functions.

For instance, any n-ary relation R on a content X defines an abstract structure J_R =(R_x, $x \in X$) modelled over R on X .

Structures of a particular order and, in that number, particularity structures are abstract.

A structure of a left regular representation of a group also is abstract.

Moreover, any structure on a set X possessing a constituent $J = (J_1 = \overline{X})$ is the abstract one. For instance, a discrete topological structure on a set X is abstract because, in this case, any element of X is its open subset.

Obviously, a universal structure on a set is abstract. It follows that any structure is represented as a constituent of an abstract structure.

3.4. Combinations of structures

Let ($I_1, \ldots, I_n, \ldots$) and ($J_1, \ldots, J_n, \ldots$) be two structures on a set X . Then their maximal common constituent ($J_1 \cap I_1, \ldots, J_n \cap I_n, \ldots$) is a structure on X called the *intersection of these structures*.

One can generalize this notion to structures ($J_1, \ldots, J_n, \ldots$) and ($I_1, \ldots, I_n, \ldots$) on different universes X and Y, respectively. For this purpose, let us consider their induced structures on an overlap $X \cap Y$ whose intersection on $X \cap Y$ is treated as an intersection of structures ($J_1, \ldots, J_n, \ldots$) and ($I_1, \ldots, I_n, \ldots$). It is defined by overlaps

$$R \cap \overset{n}{\times} Y = R \cap S = \overset{n}{\times} X \cap S$$

of domains of relations R and S which constitute structures $(J_1,\ldots,J_n,\ldots)$ and $(I_1,\ldots,I_n,\ldots)$, respectively.

In particular, it may happen that an intersection of structures is an empty structure on a non-empty overlap $X \cap Y$.

Structures are called *compatible* if their intersection is non-empty. Obviously, identically equivalent structures are compatible.

Structures are called *independent* if their intersection is empty on a non-empty overlap of their universes. For instance, two structures given only by n-ary and $(k \neq n)$-ary relations, respectively, are independent. Obviously, structures are independent if an overlap of their contents is empty.

Let $(J_1,\ldots,J_n,\ldots)$ and $(I_1,\ldots,I_n,\ldots)$ be structures on X and Y, respectively. Let us consider their extension onto a union $X \cup Y$. Then a set $X \cup Y$ is endowed with a structure $(J_1 \cup I_1,\ldots,J_n \cup I_n,\ldots)$ named the *union of original structures*.

A structure on a set is called *connected* if it is not a union of independent structures.

Clearly, a structure whose elements are non-empty n-ary and $(k \neq n)$-ary relations is disconnected. Consequently, any structure $(I_1,\ldots,I_n,\ldots)$ is a disconnected union of structures (I_1), …, (I_n),…

4. Universal structure

As was mentioned above, one can extend a notion of a structure on a set to that on a class.

Let U be a *proper class* of all sets. Then its products $U^n = \overset{n}{\times} U$, $n = 1,\ldots$, also are classes. Recall that any class is bijective to a proper class. All subsets of U^n form a proper class $V_n \approx U$. Therefore let us provide U with a structure $J_U = (V_1,\ldots,V_n,\ldots)$ generated by all subsets of classes U^n. We call it the *universal structure* on an *absolute universe* U. Then a *structure on a class* U is defined as a constituent $S = (S_1 \subset V_1,\ldots,S_n \subset V_n,\ldots)$ of a universal structure.

Let X be a set provided with a structure J. Let $\varphi: X \to U$ be its canonical bijection onto a subset $\overline{X} \subset U$ of a class U which consists of elements $\{x\} \in U$, $x \in X$. Then φ yields an isomorphic structure $\varphi(J)$ on $\overline{X} \subset U$ which is a constituent of a universal structure on U.

Consequently, any structure on a set admits an exact representation on a proper class U and, thus, it is isomorphic to a constituent of a universal structure.

Certainly, structures on sets fail to exhaust all constituents of a universal structure because their contents are only subsets of U.

Given the canonical correspondence of elements $x \in U$ to subsets $\{x\} \subset U$, any object of a universal structure is represented by its element so that an absolute universe can be identified to a constituent of a universal structure. Consequently, a universal structure is defined on its elements and, thus, is an abstract structure called the *abstract universal structure*.

It follows that any structure is represented by a constituent of an abstract structure which the abstract universal one.

In conclusion, let us note that though the fact that a class can be a domain of a relation may motivate us to generalize a notion of the structure to the case of such kind relations, we can not consider a system of relations if at least one of them is a class.

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