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GEOPHYSICS

Modeling of the Dissipation Rate of Turbulent Kinetic Energy

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Abstract—We consider a relaxation equation for turbulence wavenumber in semi-empirical turbulence closures. It is shown that the well-known phenomenological equation for the dissipation rate of turbulent kinetic energy can be related to this relaxation equation as a close approximation of the latter for stably stratified quasi-stationary flows. The proposed approach makes possible clarification of turbulent closures in the boundary layers of the atmosphere and ocean by specifying the equilibrium states and relaxation relations consistent with the direct and large eddy simulation data.

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(1) In large-scale models of the atmosphere and ocean, turbulence in the boundary layers is described using semi-empirical closures. The most common approach considers the equations for the second-order moments of hydrodynamic fields. Under the assumptions of Kolmogorov [1], the problem is simplified and requires determination of the dissipation rate ε of turbulence kinetic energy (TKE) E or, equivalently, finding turbulent length $l_T = E^{3/2}/\varepsilon$ or time scale $t_T = E/\varepsilon$. Modern closures still unsatisfactorily reproduce stably stratified turbulence [2] and diurnal cycle dynamics [3], and the parametrization of dissipative processes remains one of the main challenges.

Let us consider the traditional two-equation closures [4] for horizontally homogeneous turbulence, containing, in addition to the equations for the first-order moments, prognostic equations for E and ε:

$$\frac{\partial E}{\partial t} - \frac{\partial}{\partial z}\frac{K_m}{\sigma_E}\frac{\partial E}{\partial z} = P + B - \varepsilon, \qquad (1)$$

$$\frac{\partial \varepsilon}{\partial t} - \frac{\partial}{\partial z}\frac{K_m}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial z} = \frac{\varepsilon}{E}(C_{1\varepsilon}P - C_{2\varepsilon}\varepsilon + C_{3\varepsilon}B), \qquad (2)$$

where P is the shear production of turbulence and B is the production or consumption of energy by buoyancy forces. The coefficients of turbulent viscosity K_m and diffusivity K_h are related to E and ε by the following expressions: $K_{m,h} = S_{m,h}E^2/\varepsilon$, where $S_{m,h}$ are dimensionless stability functions. We restrict ourselves to the so-called "standard" $E-\varepsilon$ model, for which $S_m = 0.09$ and $S_h = 0.11$ are constants, and the turbulent Prandtl number $\mathrm{Pr}_t = K_m/K_h = S_m/S_h \approx 0.8$ is fixed as are the Schmidt numbers: $\sigma_E = 1$, $\sigma_\varepsilon = \mathrm{const}$.

Equation (2) is usually considered as completely empirical. It is written similarly to Eq. (1) by adding dimensional factors to each of the terms on the right-hand side. This equation contains four constants: $C_{1\varepsilon}$, $C_{2\varepsilon}$, $C_{3\varepsilon}$, and σ_ε. The simulation results and the mathematical properties of the system of equations depend strongly on the choice of constants (see [5]). The greatest uncertainty is the choice of constant $C_{3\varepsilon}$. For example, in stably stratified conditions, the values are considered in the interval $-1.0 < C_{3\varepsilon} < 1.5$. Constants $C_{1\varepsilon} \approx 1.44$ and $C_{2\varepsilon} \approx 1.92$ are determined from laboratory experiments [6]. The value $\sigma_\varepsilon \approx 1.11$ is prescribed based on the requirement of consistency of equations with the approximation of the logarithmic layer [4]. There are theoretical considerations [4, 5, 7] regarding the choice of constants, which, however, do not give an unambiguous conclusion about their universality.

We note that if the turbulent scale l_T is known, then there is no need in additional prognostic equation. For

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stratified turbulent boundary layers in the states close to statistically stationary, this scale can be determined using the Monin–Obukhov similarity theory [8] and the empirical results of its generalization to the cases where the fluxes are no longer constant with height. Equation (2) is used to describe the transition regimes with a rapid change in external parameters specifying together with Eq. (1) the evolution of a specific dimensional integral characteristic of the fluctuation scale at the moments of adjustment of the entire turbulence spectrum to a new equilibrium state.

The turbulent wave number is a convenient integral characteristic of the spectrum $\bar{k} = \int kE(k)dk / \int Edk$. For example, for two-dimensional turbulence, the energy-weighted wave number is fundamental, being one of the invariants of an ideal two-dimensional fluid, while for the three-dimensional flows, the scale $l_T \simeq 1/\bar{k}$, where $\bar{k}$ is the wave-weighted average wavenumber over the cospectrum, is a good approximation of the Prandtl turbulent mixing length (see [9]).

Here, we will show that Eq. (2) can be considered as the equivalent of a simple equation for the turbulent wave number $k_T = 1/l_T$ containing only one empirical constant. We will establish relations between the constants $C_{1\varepsilon}$, $C_{2\varepsilon}$, $C_{3\varepsilon}$, and σ_ε and show that if the choice of these constants is consistent with Monin–Obukhov similarity theory, the results of modeling of a stably stratified boundary layer may be improved.

(2) The evolution of the turbulent scale of the wave number k_T should describe the adjustment of the spectrum to the equilibrium state. We restrict ourselves to the relaxation model of such an adjustment:

$$\frac{\partial k_T}{\partial t} = -\frac{1}{t_R}(k_T - k_T^0), \tag{3}$$

where $k_T^0 = 1/l_T^0$ and l_T^0 is the equilibrium turbulent length scale determined by the geometry of the flow and stratification. Relaxation time $t_R = C_R^{-1}E/\varepsilon$ is assumed to be proportional to the turbulent time scale t_T; C_R is a new empirical constant. A similar equation under the assumption of spectrum self-similarity was obtained from the spectral balance equation in [7] for homogeneous forced turbulence. In case of decaying homogeneous turbulence, at $k_T^0 \equiv 0$ and $t \gg 1$, the solution of system (1)–(3) has asymptotics $l_T \sim t^b$, where $b > 0$ is the exponent depending on the relaxation constant. The exponential growth of the turbulent scale with time agrees with the conclusions by Kolmogorov (see [1] and references therein). Publication [1] is a short summary of a presentation by Kolmogorov at the General Meeting of the Department of Physico-Mathematical Sciences of the USSR Academy of Sciences (on January 26–28, 1942, in Kazan). During a discussion about this presentation, L.D. Landau emphasized that "the existence of the curl of velocity in a turbulent flow is limited to a finite region of space, and the qualitatively correct equations of turbulent motion should lead to such a distribution of vortices"; that is, in real flows, the limit of the value of k_T is not equal to zero.

We use Eqs. (1) and (3) to obtain an equation for the dissipation rate $\varepsilon = E^{3/2}k_T$:

$$\frac{\partial \varepsilon}{\partial t} - \bar{D}_\varepsilon = \frac{3}{2}\frac{\varepsilon}{E}(P + B - \varepsilon) + C_R\frac{\varepsilon}{E}(\varepsilon^0 - \varepsilon), \tag{4}$$

where $\varepsilon^0 = E^{3/2}k_T^0$ is the equilibrium state and operator $\bar{D}_\varepsilon$ is written as

$$\bar{D}_\varepsilon = \frac{3}{2}\frac{\varepsilon}{E}\left(\frac{\partial}{\partial z}\frac{K_m}{\sigma_E}\frac{\partial E}{\partial z}\right) \equiv \frac{3}{2}\frac{\varepsilon}{E}D_E. \tag{5}$$

Equation (4) is also a relaxation relation, in which ε is related both to the local equilibrium with the total production of TKE $P + B$ and to ε^0.

Let us define the equilibrium dissipation rate for horizontally homogeneous turbulence under stable stratification when dimensionless velocity gradient Φ_u at any z/L is approximated with a good accuracy by a universal function [10]:

$$\Phi_u \equiv \frac{\kappa z}{\tau^{1/2}}\frac{dU}{dz} = \left(1 + C_u\frac{z}{L}\right). \tag{6}$$

Here, τ is the momentum flux, $L = -\tau^{3/2}/B$ is the Obukhov length scale, $C_u = \kappa/\mathrm{Ri}_f^\infty$, κ is the Karman constant, and Ri_f^∞ is the maximum value of the Richardson flux number $\mathrm{Ri}_f = -B/P$. Then, following [11], assuming locality of turbulent processes and neglecting the third-order transport terms, we get the following expression for the equilibrium dissipation rate based on the balance of production and consumption of TKE:

$$\varepsilon^0 = P - \frac{\tau^{3/2}}{L} = \frac{\tau^{3/2}}{\kappa z}\left(1 + C_\varepsilon\frac{z}{L}\right), \tag{7}$$

where $C_\varepsilon = \kappa(1 - \mathrm{Ri}_f^\infty)/\mathrm{Ri}_f^\infty$. Universal relation (7) is confirmed by the results of direct numerical simulation and the data of measurements (see [11]) in a wide range of values of z/L including conditions of strong stability.

Taking into account relation (7) for ε^0, we write Eq. (4) as

$$\frac{\partial \varepsilon}{\partial t} - \bar{D}_\varepsilon - C_R\frac{\varepsilon}{E}\frac{\tau^{3/2}}{\kappa z} = \frac{\varepsilon}{E}(\bar{C}_{1\varepsilon}P + \bar{C}_{3\varepsilon}B - \bar{C}_{2\varepsilon}\varepsilon), \tag{8}$$

where

$$\begin{gathered}\bar{C}_{1\varepsilon} = 3/2,\\ \bar{C}_{2\varepsilon} = 3/2 + C_R,\\ \bar{C}_{3\varepsilon} = 3/2 - C_R(1 - \mathrm{Ri}_f^\infty)/\mathrm{Ri}_f^\infty.\end{gathered} \tag{9}$$

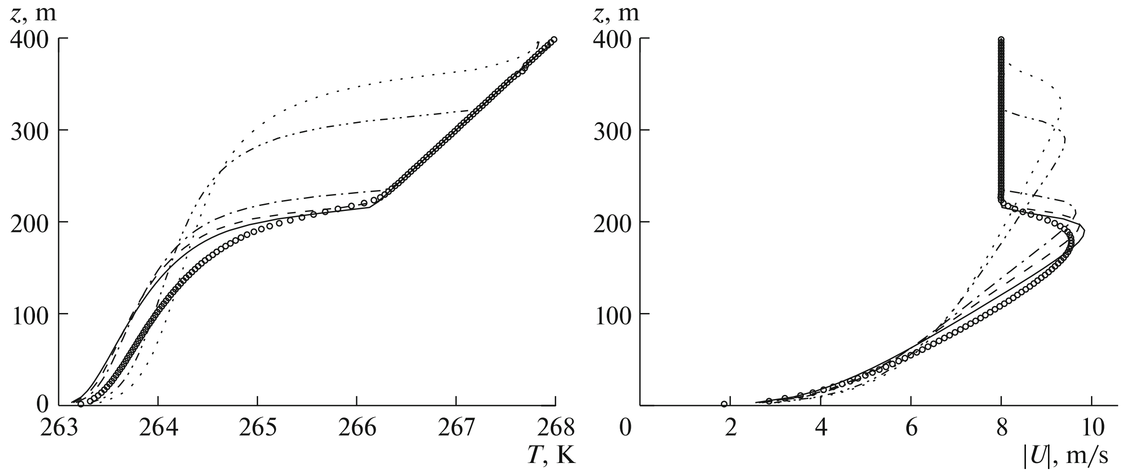

Fig. 1. Vertical distribution of the potential temperature (left) and the wind speed (right). The relaxation equation for the dissipation rate (4): $C_R = 0.48$, solid line. Standard equation (2): $C_{3\varepsilon} = -0.4$, dashed line; $C_{3\varepsilon} = 0$, dashed–dotted line; $C_{3\varepsilon} = 1.14$, dashed–double dotted line; $C_{3\varepsilon} = 1.44$, dotted line. LES-model: circles.

The standard equation for the dissipation rate (2) can be considered as a particular case of equation (8), in which the last two terms on the left-hand side can be approximated by the diffusion operator:

$$\bar{D}_\varepsilon + C_R \frac{\varepsilon}{E}\frac{\tau^{3/2}}{\kappa z} \approx D_\varepsilon \equiv \frac{\partial}{\partial z}\frac{K_m}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial z}. \tag{10}$$

We limit ourselves to the neutral stratification and consider only the surface layer with constant fluxes, in which the TKE value depends only slightly on height; hence, $D_E \approx 0$ and $|\bar{D}_\varepsilon| \ll C_R \frac{\varepsilon}{E}\frac{\tau^{3/2}}{\kappa z}$. Then constant σ_ε can be expressed by means of constant C_R as follows:

$$\sigma_\varepsilon = \frac{\kappa^2}{S_m^{1/2} C_R} = \frac{\kappa^2}{S_m^{1/2}(\bar{C}_{2\varepsilon} - \bar{C}_{1\varepsilon})}. \tag{11}$$

A similar expression for constant σ_ε can also be obtained in the standard $E-\varepsilon$ model (1)–(2), by taking into account the asymptotics $l_T \sim z$ for the length scale near the surface [4].

Thus, it can be expected that, under neutral and weakly stable stratification, the standard equation for the dissipation rate (2) and Eq. (4) obtained from relaxation equation (3) will have close stationary solutions if the equilibrium dissipation ε^0 is determined from a relation of type (7) and the set of constants $C_{p\varepsilon} \approx \bar{C}_{p\varepsilon}$ $(p = 1, 2, 3)$ and σ_ε satisfy relations (9), (11).

However, in the case of nonstationary regimes, the dynamics of the systems under consideration may turn out to be different due to the poor representability of the residual terms in Eq. (8) in the form of a diffusion operator.

The best agreement between the equilibrium solution (7) and the results of direct numerical simulation [11, 12] are observed at $\mathrm{Ri}_f^\infty \approx 0.2$, $\kappa \approx 0.4$. We use these values and relations (9) at $C_{2\varepsilon} - C_{1\varepsilon} = 0.48$ and obtain $C_{3\varepsilon} \approx -0.4$. This selection of constant $C_{3\varepsilon}$ is the most justified for the standard $E-\varepsilon$ model.

(3) We used the formulation of numerical experiments [2], proposed in the framework of the Global Energy and Water Exchanges, Atmospheric Boundary Layer Study program (GEWEX GABLS), to compare turbulence models for reproducing a height-increasing, stably-stratified atmospheric boundary layer. The constants given in Section 1 were used in the standard model (1)–(2), while the value of $C_{3\varepsilon}$ varied. Experiments with relaxation equation (4) were performed at $C_R = 0.48$.

Figures 1 and 2 show the profiles of temperature, wind speed, kinematic turbulent heat flux, and the total momentum flux averaged over the ninth hour of calculation, in comparison with the large eddy simulation (LES) data [13]. At $C_{3\varepsilon} \approx -0.4$ the solution of the standard model is close to the results of the model with relaxation equation (4) and to the LES data. Deviations are comparable with the discrepancies in the results of large eddy simulations [14]. When the selection of constant $C_{3\varepsilon}$ is inconsistent, the height of the boundary layer in the standard model is overestimated and the error in reproducing turbulent heat and momentum fluxes increases.

(4) The equation for the turbulence kinetic energy dissipation rate ε was obtained from the equation of relaxation of the turbulent wave number k_T to its equilibrium value. It is shown that the derived equation has a form close to that of the standard phenomenological equation for the dissipation rate. A relationship is established between the four constants in the standard equation for the dissipation rate and the relaxation

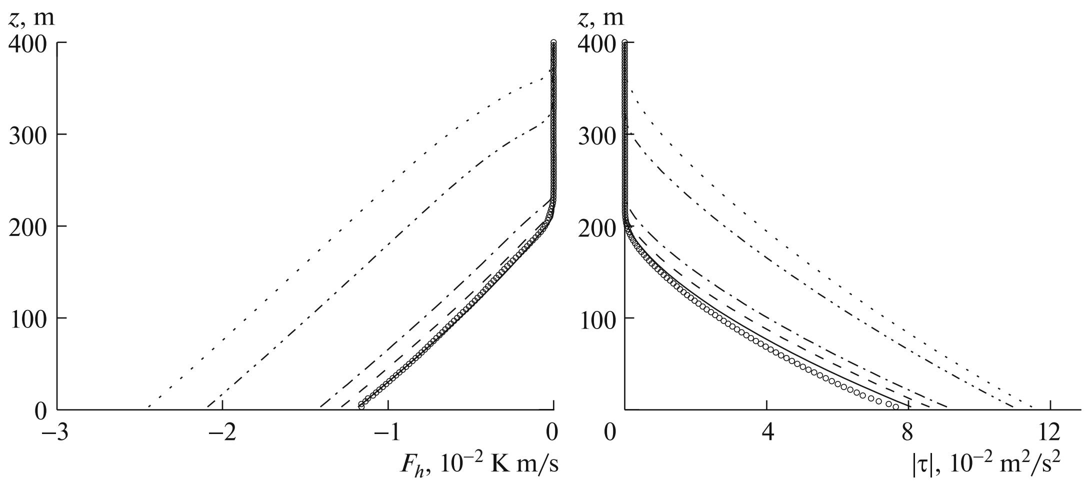

Fig. 2. Vertical distribution of the heat flux (left) and total momentum flux (right). The designations of the lines coincide with those shown in Fig. 1.

constant C_R in the equation for k_T. It is shown that when the equilibrium state of the system is consistent with local Monin–Obukhov similarity scaling, and the choice of constants is appropriate, both approaches considered here lead to close results. Moreover, the basic characteristics of stably stratified boundary layers near steady state are reproduced well.

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