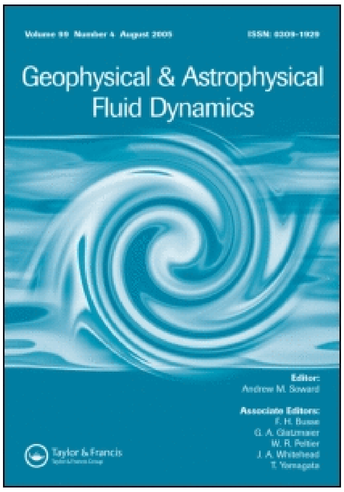

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Large-scale vortices in helical turbulence of incompressible fluid

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LARGE-SCALE VORTICES IN HELICAL TURBULENCE OF INCOMPRESSIBLE FLUID

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The interaction of a mean flow with a random fluctuation field is considered. This interaction is described by the averaged Navier-Stokes equation in which terms nonlinear in the fluctuation field are expressed in terms of the mean flow and the statistical properties of the fluctuation field, which is assumed to be homogeneous, isotropic, and helical. Averaged equations are derived using a functional technique. These equations are solved for a mean background flow that depends linearly on the position vector. The solutions show that large-scale vortices may arise in this system.

1. INTRODUCTION

The problem of the development of large-scale, long-lived vortices in turbulence, that are usually called coherent structures, has recently attracted much attention. Here we consider those coherent structures that are generated by the turbulence rather than those that are produced by the decay of regular flows or are due to instability of a background flow (e.g., the Karman vortices). In statistical theories, the turbulent medium is usually considered to be homogeneous and isotropic. The questions arise: is self-organization possible in turbulent media, and under what conditions do coherent structures arise? On the one hand, common sense suggests that it is very difficult to extract energy from a chaotic system, and only some specific additional properties of such systems can make it possible. On the other hand, we frequently see evidence of self-organization in nature. For instance, it is widely believed that tropical cyclones (typhoons) extract their energy from small-scale convection; there are no regular flows in tropics that can feed a typhoon through their decay. It is clear from thermodynamical reasoning that homogeneous, isotropic turbulence, which does not possess any distinguished scales or preferred directions, is too symmetric to give birth to large-scale vortices. Self-organization seems to be improbable in this case. Thus, we expect the breaking of some symmetry to be a necessary condition for self-organization.

In this paper we consider turbulence with broken mirror symmetry. A simple example of such a turbulence is a helical one. In helical turbulence, the correlation $\langle \mathbf{V}' \cdot \nabla \times \mathbf{V}' \rangle$ is nonzero, where $\mathbf{V}'$ is the random fluctuation velocity; the numbers of

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right-handed and left-handed vortices are not equal to each other. The integral $\int\langle \mathbf{V}' \cdot \times \mathbf{V}'\rangle d^3r$ is a topological invariant of the motion that characterises the number of linkages of streamlines in the fluid volume.

The hypothesis that a helical turbulence can amplify large-scale vortices was proposed rather long ago on the basis of the similarity between the equations governing vorticity in hydrodynamics and magnetic fields in magnetohydrodynamics (see Moffatt, 1981 and references therein). It was shown by Steenbeck, Krause and Rädler (1966) (see also Krause and Rädler, 1980) that helical turbulence can amplify large-scale magnetic fields in MHD even when it is homogeneous and isotropic. This phenomenon is usually called the alpha-effect. It is a common point of view that the magnetic fields of spiral galaxies and of the Sun can be explained on the basis of the alpha-effect (see, e.g., Parker, 1979; Zeldovich, Ruzmaikin, and Sokoloff, 1983). Therefore, an attempt to establish an analoge for the alpha-effect in hydrodynamics seems to be quite natural.

Krause and Rüdiger (1974) have shown that in an incompressible fluid the effect analogous to the alpha-effect is precluded by the symmetry of the mean-field Reynolds stresses in the averaged Navier-Stokes equation. Their result seems to forbid completely a hydrodynamical counterpart of the alpha-effect. The analogy between the vorticity and induction equations thus seems to be a purely formal one, because the nonlinearity of the vorticity equation finds no parallel in the linearity of the induction equation.

Nevertheless, the generation of large-scale vortices by helical turbulence is a real effect. This statement does not contradict the conclusions of Krause and Rüdiger (1974). As a matter of fact, the presence of helicity alone is insufficient in hydrodynamical turbulence for energy transfer from small scales to large ones; some additional symmetry violation is necessary.

As shown by Moiseev, *et al.* (1983a) (see also Tur *et al.* 1984; Sagdeev *et al.* 1984), a hydrodynamical alpha-effect exists in compressible fluids, where the role of the additional symmetry-breaking factor is played by compressibility. Hydrodynamic alpha-effect equations have been derived there under the assumption that the turbulence is a random process δ-correlated in time. The form of the final equations for the mean vorticity coincides exactly with the MHD α-effect equations. The pseudoscalar coefficient α in the generation term $\nabla\times(\alpha\nabla\times\mathbf{V})$ proved to be proportional to the helicity of the turbulence and independent of the compressibility parameter, $M=\lambda_{\mathrm{cor}}/(c\tau_{\mathrm{cor}})$, where λ_{cor} are spatial and temporal characteristic scales of the turbulence, and c is the sound speed. We would like to clarify this point. If one abandons a δ-correlated approximation the factor α becomes dependent on M. When $\tau_{\mathrm{cor}}\ll(\lambda_{\mathrm{cor}}/c)$ i.e. $M\gg 1$, the leading term in the asymptotic expansion of $\alpha(M)$ in powers of M^{-2}, is independent of M. The limit $M\to\infty$ as $\tau_{\mathrm{cor}}\to 0$, that corresponds to the δ-correlated approximation, is considered by Moiseev *et al.* (1983a). In the oppostie limit $\tau_{\mathrm{cor}}\gg(\lambda_{\mathrm{cor}}/c)$, i.e. $M\ll 1$, the factor α is proportional to M^2.

We show below (see also Tur *et al.*, 1987) that the hydrodynamic alpha-effect is present also in incompressible fluids when the helical turbulence is supplemented by a large-scale flow, which breaks the symmetry. Sagdeev *et al.* (1985) have given another example of the alpha-effect in an incompressible liquid in which a temperature gradient in the gravity field serves as a symmetry breaking factor. These authors and Moiseev *et al.* (1986, 1987, 1988) have shown that the alpha-effect is present in helical turbulent convection.

The connection between helicity and inverse energy cascades in hydrodynamical turbulence has been noted by Brissaud *et al.* (1973) and Kraichnan (1973). Indeed, the nonlinear term in the vorticity equation can be expressed as $\nabla \times [(\nabla \times \mathbf{V}) \times \mathbf{V}]$. When the directions of $\nabla \times \mathbf{V}$ and $\mathbf{V}$ are correlated, the inertia term vanishes and Kolmogorov's energy cascade toward small scales is suppressed. The connection between helicity fluctuations and inverse energy cascade is discussed also by Levich and Tzvetkov (1985). Recently Frisch *et al.* (1987) have given a new and important example of large-scale vortex generation in an incompressible fluid. Their approach is based on averaging over the spatial periodicity of a three-dimensional regular anisotropic flow lacking parity invariance: this leads to the growth of large-scale modes, an anisotropic alpha-effect.

In this paper we derive and analyse an averaged equation that describes the generation of large-scale vortices by helical turbulence. In our approach we apply a closure procedure to the averaged equations, which is based on a functional technique; we do not use any model closure hypothesis. The paper is organized as follows. In Section 2 the basic equations are derived, and qualitative aspects are considered. In Section 3 we derive the averaged equation that describes the evolution of large-scale perturbations under the influence of helical turbulence and steady nonuniform large-scale flow. In Section 4 we solve that equation in the particular case a large-scale potential flow having components linear in the coordinates. Concluding remarks are presented in Section 5. It is shown that the averaged equation describes the development of large-scale vortices in helical turbulence.

2. BASIC EQUATIONS AND QUALITATIVE CONSIDERATIONS

We start from the equations of motion of an incompressible fluid

$$\partial_t V_i + V_k\, \partial_k V_i = \nu_0 \Delta V_i - \rho^{-1}\, \partial_i P, \tag{2.1}$$

$$\partial_i V_i = 0, \tag{2.2}$$

where V_i is the velocity field, P is the pressure (the density ρ is uniform), ν_0 is the kinematic viscosity. Using the incompressibility condition, we eliminate the pressure to obtain

$$\partial_t V_i + \Pi_{ij} V_k\, \partial_k V_j = \nu_0 \Delta V_i, \tag{2.3}$$

where $\Pi_{ij} = \delta_{ij} - \Delta^{-1}\, \partial_i \partial_j$ is the projection operator, and Δ^{-1} is the inverse Laplace operator. Represent the velocity field as a sum of regular and random components:

$$\mathbf{V} = \langle \mathbf{V} \rangle + \mathbf{V}', \qquad \langle \mathbf{V}' \rangle = 0, \tag{2.4}$$

where angular brackets denote the ensemble average. The averaged Navier-Stokes equation has the form

$$\partial_t\langle V_i\rangle+\Pi_{ij}\,\partial_k(\langle V_k\rangle\langle V_j\rangle+\langle V'_kV'_j\rangle)=\nu_0\Delta\langle V_i\rangle. \tag{2.5}$$

The incompressibility condition $\partial_i\langle V_i\rangle=0$ is always implied. The averaged equation (2.5) includes an unknown term $\langle V'_iV'_k\rangle$ which represents the Reynolds stresses, and closure is required. Our aim is to express the Reynolds stresses in terms of the mean field $\langle \mathbf{V}\rangle$ and given statistical parameters of the turbulence. Thus, we consider the influence of the turbulence on mean fields and do not take into account the back reaction of the latter on the turbulence. From the formal point of view, this means that our aim is to derive a mean-field differential equation that involves statistical parameters of the turbulence as coefficients. We are especially interested in the generation and evolution of large-scale vortices, i.e. vortices whose spatial and temporal scales, L and T, are large compared with the energy-range scales of the turbulent fluctuations, λ and τ. Thus, the statement of the problem implies the presence of two small parameters. These are the ratios of spatial and temporal scales, $\lambda/L\ll 1$ and $\tau/T\ll 1$. The two-scale approximation seems to be justified since we are concerned with the influence of small-scale motions on the large-scale ones, rather than with inverse energy cascades or other statistical turbulent properties.

Thus, consider a given homogeneous isotropic turbulence with non-zero mean helicity. The correlation tensor of this turbulent flow has the form:

$$\langle V'_i(\mathbf{r}_1)V'_j(\mathbf{r}_2)\rangle=C(r)\delta_{ij}+B(r)r_ir_j+g(r)\varepsilon_{ijm}r_m, \tag{2.6}$$

where $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, δ_{ij} is the unit tensor, ε_{ijm} is the unit, completely antisymmetric tensor, $C(r)$ and $B(r)$ are scalar functions and, $g(r)$ is a pseudoscalar function. Expression (2.6) is the most general form of the two-point correlation function for a homogeneous, isotropic, helical vector field (see Batchelor, 1953; Monin and Yaglom, 1965). Indeed, since the vector field is homogeneous, the correlation function can depend only on the difference between $\mathbf{r}_1$ and $\mathbf{r}_2$. Due to isotropy, the scalar functions C and B and pseudoscalar function g can depend only on the modulus of the vector $\mathbf{r}$; in constructing the correlation tensor we can use only one vector $\mathbf{r}$ and two constant tensors, δ_{ij} and ε_{ijk}. Thus, we arrive at expression (2.6). Note that ε_{ijk} is a pseudotensor but the correlation $\langle V'_iV'_j\rangle$ is a real tensor, so that $g(r)$ should be a pseudoscalar. The function g has a simple physical meaning: $\langle \mathbf{V}'(\mathbf{r})\cdot\nabla\times\mathbf{V}'(\mathbf{r})\rangle=6g(0)$. Indeed, the quantity $\langle \mathbf{V}\cdot\nabla\times\mathbf{V}\rangle$ is a pseudoscalar just like g. The scalar product $\mathbf{V}\cdot\nabla\times\mathbf{V}$ is usually called the helicity. The helicity integral over a liquid volume, $\int(\mathbf{V}\cdot\nabla\times\mathbf{V})d^3r$ is a topological invariant (an integral of motion in ideal fluids) that characterizes the topology of the vector field $\mathbf{V}$ lines (see Moffatt, 1981 and references therein). Helical turbulence is an example of turbulence lacking mirror symmetry. This implies that the right-hand and left-hand systems of coordinates can be distinguished through measurements of certain parameters ($\langle \mathbf{V}\cdot\nabla\times\mathbf{V}\rangle$ in a helical turbulence).

Let us assume now that we have a large-scale velocity fluctuation, $\langle \mathbf{V}\rangle$, superimposed on a homogeneous, isotropic, helical turbulence. This fluctuation interacts with the turbulence, and the latter acquires an inhomogeneous perturbation. The random velocity field thus can be decomposed into two parts, homogeneous and inhomogeneous fields: $\mathbf{V}' = \mathbf{V}^t + \tilde{\mathbf{V}}$. Now expression (2.6) describes the correlation function of the homogeneous field. $\langle \mathbf{V}^t\mathbf{V}^t \rangle$, while an inhomogeneous perturbation to the correlation function has the form $\langle \mathbf{V}^t\tilde{\mathbf{V}} \rangle$, where we have neglected the nonlinear perturbation $\langle \tilde{\mathbf{V}}\tilde{\mathbf{V}} \rangle$ (such an omission is justified when the Reynolds number is small). In other words, we consider a weakly nonlinear regime. This yeidls the following equation:

$$\partial_t \langle V_i \rangle + \Pi_{ij}\, \partial_k (\langle V_j^t \tilde{V}_k \rangle + \langle \tilde{V}_j V_k^t \rangle + \langle V_j \rangle \langle V_k \rangle) = \nu_0 \Delta \langle V_i \rangle, \tag{2.7}$$

where the term $\Pi_{ij}\, \partial_k \langle V_j^t V_k^t \rangle$ vanishes due to homogeneity of the random field $\mathbf{V}^t$.

Now the question is: what forms of the nonlinear term in (2.7) are admissible? Recall that $\langle \mathbf{V}^t\tilde{\mathbf{V}} \rangle$ is inhomogeneous. But there is only one inhomogeneous quantity in our problem, viz. $\langle \mathbf{V}(\mathbf{r}) \rangle$. Therefore, the unknown tensor $\langle \mathbf{V}^t\tilde{\mathbf{V}} \rangle$ must be a function of $\langle \mathbf{V} \rangle$, which can be expanded in a series in power of $\langle \mathbf{V} \rangle$ and its derivatives,

$$\langle V_i^t \tilde{V}_j \rangle = T_{ij}^{(0)} + T_{ijl}^{(1)} \langle V_l \rangle + T_{ijlm}^{(2)}\, \partial_l \langle V_m \rangle + \ldots \tag{2.8}$$

where the tensors $T^{(n)}$ are constants:

$$T_{ijl}^{(1)} \sim \left. \frac{\partial \langle V_i^t \tilde{V}_j \rangle}{\partial \langle V_l \rangle} \right|_{\langle V_l \rangle = 0}, \quad \text{etc.}$$

The tensors $T^{(n)}$ can be constructed from the tensors δ_{ik}, ε_{ijk}, the scalar C and the pseudoscalar g. Since (2.5) involves the spatial derivatives of the Reynolds stresses, the constant tensor $T^{(0)}$ makes no contribution to (2.5). The third rank tensor $T^{(1)}$ can only be ε_{ikl}. (Obviously, a third rank tensor cannot be constructed from δ_{ik} and scalar constants.) But (2.5) involves a combination of tensors that is symmetric in its indices, namely $\langle V_i^t \tilde{V}_j \rangle + \langle V_j^t \tilde{V}_i \rangle$. Therefore, the term proportional to $T^{(1)}$ also vanishes in (2.5). Thus, the alpha-term (that has the form $\nabla \times (\alpha \mathbf{V})$, with α proportional to the mean helicity) is precluded in the averaged Navier-Stokes equation (Krause and Rüdiger, 1974). This brings us to Reynolds hypothesis, which presumes that

$$\partial_j (\langle V_i^t \tilde{V}_j \rangle + \langle V_j^t \tilde{V}_i \rangle) = -\nu_t \Delta \langle V_j \rangle,$$

where ν_t is the turbulent viscosity. It can be easily shown that the incompressibility condition $\partial_i V_i = 0$ leads to

$$T_{ijkl}^{(2)}\, \partial_l\, \partial_k \langle V_j \rangle = \text{const}\, \Delta \langle V_i \rangle.$$

The constant coefficient ν_t has been evaluated by Krause and Rüdiger (1974).

It is now clear that the alpha-effect in hydrodynamics is possible either when the nonlinear term in equation (2.5) does not possess the above mentioned symmetry (e.g., in compressible fluids; see Moiseev *et al.*, 1983a) or when the tensor $T^{(1)}$ includes some additional terms apart from the tensor ε_{ikl}. The latter case is possible if, for example, a three-dimensional steady flow, $\mathbf{V}^{(0)}(\mathbf{r})$, is present. We do not discuss here the possible forms of the tensors $T^{(n)}$ in this case, since we evaluate them explicitly below. We note only that as a result the nonlinear term in (2.5) not only affects the viscosity, but also gives new terms proportional to the first derivatives of $\langle \mathbf{V} \rangle$ with respect to the coordinates,

$$\partial_k(\langle V_i^t \tilde{V}_k \rangle + \langle V_k^t \tilde{V}_i \rangle) = G_{ikl}\, \partial_k \langle V_l \rangle - \nu_T \Delta \langle V_i \rangle, \tag{2.9}$$

where the tensor G depends on both the steady flow $\mathbf{V}^{(0)}$ and the correlation tensor of the turbulence. It seems that by invoking a regular flow we make the problem less interesting, since it is clear that large-scale vortices can originate in such systems through the decay of this regular flow. However, we show below that, in our case, large-scale vortices arise even when $\mathbf{V}^{(0)}$ is a *potential* flow. This fact allows us to understand the symmetry-breaking role of the regular flow as a trigger mechanism that removes obstacles to the generation of large-scale structures by helical turbulence.

Let us now redefine the background state. Represent the basic flow $\mathbf{V} = \mathbf{V}^{(0)} + \mathbf{V}^t$ as a combination of a regular steady flow $\langle \mathbf{V} \rangle = \mathbf{V}^{(0)}$ and a statistically-steady (stationary) turbulence $\mathbf{V}^t$ that are described by

$$\partial_t V_i^t + \Pi_{ij}\, \partial_k (V_j^{(0)} V_k^t + V_j^t V_k^{(0)} + V_j^t V_k^t) = \nu_0 \Delta V_i^t + \Pi_{ij}\, \partial_k \langle V_j^t V_k^t \rangle + F_i', \tag{2.10}$$

$$\Pi_{ij}\, \partial_k (V_k^{(0)} V_j^{(0)} + \langle V_k^t V_j^t \rangle) = \nu_0 \Delta V_i^{(0)} + F_i^{(0)}, \tag{2.11}$$

where $\mathbf{F}$ is the external force, $\mathbf{F} = \mathbf{F}^{(0)} + \mathbf{F}'$ (with $\mathbf{F}^{(0)} = \langle \mathbf{F} \rangle$) that drives both the steady flow $\mathbf{V}^{(0)}$ and the turbulence $\mathbf{V}^t$. The turbulence $\mathbf{V}^t$ is supposed to be homogeneous, isotropic and helical; the correlation function $\langle V_i^t V_j^t \rangle$ is given by (2.6). Since the turbulence is produced by an external force, we may write $(\langle \mathbf{V}^t \mathbf{V}^t \rangle^{1/2})/V^{(0)} \sim \varepsilon$, where $\varepsilon \ll 1$. In addition, we assume that the turbulent Reynolds number is small.

Now we disturb the basic flow,

$$\mathbf{V} = (\mathbf{V}^{(0)} + \mathbf{V}^t) + (\mathbf{V}^{(1)} + \tilde{\mathbf{V}}), \tag{2.12}$$

where $\mathbf{V}^{(1)}$ is the perturbation of the regular component and $\tilde{\mathbf{V}}$ is the perturbation of the random component of the velocity field. Suppose that, because of a presumed low value of the Reynolds number, these perturbations are small. Thus, we may envisage the following scenario. In the basic flow, that is described by (2.10) and (2.11), a large-scale fluctuation $\mathbf{V}^{(1)}$ arises. This fluctuation interacts with the turbulence and evolves in time so that we may consider $\tilde{\mathbf{V}}$ as a response of the turbulence to the disturbance $\mathbf{V}^{(1)}$. Thus, our problem can be understood as a stability problem for the basic flow.

On neglecting the nonlinear perturbation $\langle \tilde{\mathbf{V}} \tilde{\mathbf{V}} \rangle$, the equation for the large-scale fluctuation $\mathbf{V}^{(1)}$ takes the form:

$$\partial_t V_i^{(1)} + \Pi_{ij}\, \partial_k (V_j^{(0)} V_k^{(1)} + V_k^{(0)} V_j^{(1)} + V_j^{(1)} V_k^{(1)}) = \nu_0 \Delta V_i^{(1)} - \Pi_{ij}\, \partial_k (\langle V_k^t \tilde{V}_j \rangle + \langle V_j^t \tilde{V}_k \rangle). \tag{2.13}$$

The equation for the small-scale perturbation of the turbulent velocity, $\tilde{\mathbf{V}}$, is

$$\partial_t \tilde{V}_i + \Pi_{ij}\, \partial_k (V_k^{(0)} \tilde{V}_j + V_j^{(0)} \tilde{V}_k + V_k^{(1)} V_j^t + V_j^{(1)} V_k^t) = \nu_0 \Delta \tilde{V}_i, \tag{2.14}$$

where small terms $\tilde{\mathbf{V}}\tilde{\mathbf{V}}$, $\tilde{\mathbf{V}}\mathbf{V}^t$, $\mathbf{V}^{(1)}\tilde{\mathbf{V}}$ are neglected.

Equations (2.13) and (2.14) are the basic equations of the problem. Equation (2.13) involves an unknown quantity $\langle \mathbf{V}^t \tilde{\mathbf{V}} \rangle$. Closure is achieved after we have evaluated this correlation in the next section using (2.14), which determines the random function $\tilde{\mathbf{V}}$ as a functional of the given random process $\mathbf{V}^t$.

3. CLOSURE OF THE BASIC EQUATION

To calculate one-point correlations $\langle V_k^t(\mathbf{r},t)\tilde{V}_j(\mathbf{r},\mathbf{t})\rangle$ and $\langle \tilde{V}_k(\mathbf{r},t)V_j^t(\mathbf{r},t)\rangle$ we use a functional technique (see Tatarsky, 1974; Klyatskyn, 1975). These correlations are the mean values of the products of the functional $\tilde{\mathbf{V}}$, of the random process $\mathbf{V}^t$, and the process $\mathbf{V}^t$ itself. Using the Furutsu-Novikov formula, we can express these mean values in terms of integrals of the random process correlation function and the average of the functional derivative of this functional in the random process:

$$\langle F[\phi]\phi(t)\rangle = \int \langle \phi(t)\phi(t')\rangle \left\langle \frac{\delta F[\phi]}{\delta\phi(t')} \right\rangle dt' + \ldots, \tag{3.1}$$

where $F[\phi]$ is a functional of the random process $\phi(t)$. In our case the quantity $\tilde{\mathbf{V}}$ is a functional of the random process $\mathbf{V}^t$. To obtain the required mean value we should evaluate the functional derivative $\partial\tilde{\mathbf{V}}/\delta\mathbf{V}^t$. First, we solve (2.14) and obtain an explicit expression for $\tilde{\mathbf{V}}$. Then the functional derivative can be evaluated. Recast (2.14) as

$$\partial_t V_i(\mathbf{r},t) + \hat{M}_{ik}(\mathbf{r}) \tilde{V}_k(\mathbf{r},t) = T_i(\mathbf{r},t), \tag{3.2}$$

where the operator $\hat{M}$ has the form

$$\left.\begin{aligned} \hat{M}_{ik}(\mathbf{r}) &= -\nu_0 \Delta\, \delta_{ik} + \hat{N}_{ik}(\mathbf{r}), \\ \hat{N}_{ik}(\mathbf{r}) &= \partial_m \Pi_{ij} [V_m^{(0)}(\mathbf{r})\delta_{jk} + V_j^{(0)}(\mathbf{r})\delta_{mk}], \end{aligned}\right\} \tag{3.3}$$

and the right-hand side of (3.2) is given by

$$T_i(\mathbf{r},t) = -\partial_m \Pi_{ij} (V_m^{(1)} V_j^t + V_j^{(1)} V_m^t), \tag{3.4}$$

The operator $\hat{M}$ in (3.2) does not depend on time. Thus, the solution (see Bellman, 1960) has the form

$$\tilde{\mathbf{V}}(\mathbf{r},t) = e^{-\hat{M}t}\,\mathbf{C} + \int e^{\hat{M}(t-s)}\,\mathbf{T}(s)ds, \tag{3.5}$$

where the vector $\mathbf{C}$ is determined by initial conditions and is independent of time. The initial condition $\tilde{\mathbf{V}}(\mathbf{r},0)=0$ gives $\mathbf{C}=0$. Note that the operator $\hat{M}$ in the exponent in expression (3.5) involves two noncommutative parts [see (3.3)]. Indeed, $\mathbf{V}^{(0)}(\mathbf{r})$ depends on the position, and $\hat{N}$ does not commute with the Laplace operator. However, we consider a weakly nonlinear regime (the turbulent Reynolds number is small) in which, in dimensionless variables, the second term in (3.3) is proportional to a small parameter. To expand the operator exponent in powers of a small parameter, we use the Feynman formula (see Feynman, 1951; Bellman, 1960)

$$e^{\hat{A}+\alpha\hat{B}}=e^{\hat{A}}\left[\hat{I}+\alpha\int_0^1 e^{-\hat{A}\xi}\,\hat{B}\,e^{\hat{A}\xi}\,d\xi+\dots\right] \tag{3.6}$$

where $\alpha\ll 1$, $\hat{I}$ is the unit matrix and the operators $\hat{A}$ and $\hat{B}$ do not commute. Using this formula we can represent the solution (3.5) of (3.2) as

$$\tilde{V}_i(\mathbf{r},t)=\int_0^t e^{\nu_0\Delta(t-s)}\left[\delta_{ik}-(t-s)\int_0^1 e^{-\nu_0\Delta(t-s)\xi}\hat{N}_{ik}(\mathbf{r})\,e^{\nu_0\Delta(t-s)\xi}\right]T_k(\mathbf{r},s)\,ds, \tag{3.7}$$

where terms of second and higher order are omitted. In this approximation, the dependence of the linear functional $\tilde{\mathbf{V}}$ on the random process $\mathbf{V}^t$ is given by (3.7) and the evaluation of the functional derivative becomes trivial. The result is

$$\left\langle\frac{\delta\tilde{V}_i(\mathbf{r},t)}{\delta V_n^t(\mathbf{r}',t')}\right\rangle=\tfrac{1}{2}e^{\nu_0\Delta(t-t')}\left[\delta_{ik}-(t-t')\int_0^1 e^{-\nu_0\Delta(t-t')\xi}\hat{N}_{ik}(\mathbf{r})\,e^{\nu_0\Delta(t-t')\xi}\,d\xi\right]W_{kn}(\mathbf{r},\mathbf{r}',t),$$

$$W_{kn}(\mathbf{r},\mathbf{r}',t')=-\partial_m\Pi_{kj}[V_m^{(1)}(\mathbf{r},t')\delta_{nj}+V_j^{(1)}(\mathbf{r},t')\delta_{mn}]\delta(\mathbf{r}-\mathbf{r}'). \tag{3.8}$$

(Note, that the higher functional derivatives vanish due to linearity of $\tilde{\mathbf{V}}$ in $\mathbf{V}^t$.)

The Furutsu-Novikov formula for the correlator $\langle\tilde{\mathbf{V}}\mathbf{V}^t\rangle$ gives

$$\langle\tilde{V}_i(\mathbf{r}_1,t_1)V_j^t(\mathbf{r}_2,t_2)\rangle=\int\langle V_j^t(\mathbf{r}_2,t_2)V_k^t(\mathbf{r}',t')\rangle\left\langle\frac{\delta\tilde{V}_i(\mathbf{r}_1,t_1)}{\delta V_k^t(\mathbf{r}',t')}\right\rangle dt'\,d^3r', \tag{3.9}$$

where the correlation function of the random field $\mathbf{V}^t$ is given by (2.6) and the averaged functional derivative $\langle\delta\tilde{\mathbf{V}}/\delta\mathbf{V}^t\rangle$ is determined by (3.6). Now we are ready to evaluate the integral in (3.9). To do this, expand the operators in (3.9) in powers of $\lambda/L$where λ is a small scale characteristic energy-containing range of the turbulence and L is the large scale of the mean flow. Details of calculations are given in the Appendix. The result is the following averaged equation:

$$\partial\, V_i^{(1)}+B\,\partial_k(V_i^{(0)}V_k^{(1)}+V_k^{(0)}V_i^{(1)})+\partial_k V_i^{(1)}V_k^{(1)}+G_{ijk}\,\partial_j V_k^{(1)}=\nu\Delta V_i^{(1)}-\rho^{-1}\,\partial_i P^{(1)}, \tag{3.10}$$

where

$$G_{ijk}=\tfrac{1}{2}G(5\partial_l V_i^{(0)}\varepsilon_{ljk}+5\partial_l V_m^{(0)}\varepsilon_{mlk}\,\delta_{ij}-3\partial_l V_k^{(0)}\varepsilon_{ijl}-3\partial_j V_l^{(0)}\varepsilon_{ilk}). \tag{3.11}$$

$P^{(1)}$ is the mean pressure, $\nu = \nu_0 + \nu_t$, with

$$\nu_t = \frac{1}{15\pi^2} \int_0^\infty \int_0^\infty dq\, d\tau\, e^{\nu_0 q^2 \tau}(q^2 + \nu_0 q^4 \tau) C(q,\tau), \tag{3.12}$$

[Expression (3.12) for ν_t coincides with result of Krause and Rüdiger, 1974.] The constants G and B are given by

$$G = \frac{1}{15}\left(\frac{2}{r}\right)^{1/2} \int_0^\infty \int_0^\infty dq\, d\tau\, q^4 \tau\, e^{\nu_0 q^2 \tau} g(q,\tau), \tag{3.13}$$

$$B = 1 - \frac{1}{15}\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \int_0^\infty dq\, d\tau\, q^4 \tau\, e^{-\nu_0 q^2 \tau}\, C(q,\tau), \tag{3.14}$$

where $g(q,\tau)$ and $C(q,\tau)$ are coefficients in the Fourier transformation of the correlation tensor (2.6) (see the Appendix). Note, that the constant G is proportional to the mean helicity of the basic turbulent velocity field, while ν_t is proportional to mean energy of the turbulence. The constant B is proportional to the mean vorticity.

Equation (3.10) is the main equation of this work. It has been derived from the closure of (2.13). The closure procedure depends on the smallness of three parameters, λ/L, τ/T, and the turbulent Reynolds number. Equation (3.10) describes the evolution of a large-scale perturbation $\mathbf{V}^{(1)}$ in a stationary, homogeneous, isotropic and helical turbulence. The coefficients G_{ijk}, B and ν in (3.10) are expressed in terms of statistical parameters of the turbulence and the components of the nonuniform basic flow $\mathbf{V}^{(0)}$. The term involving the first derivative of $\mathbf{V}^{(1)}$ with respect to the coordinates is quite similar to the alpha-term in mean-field MHD. As we show below, it is this term that is associated with generation of large-scale vortices. Its presence is due to the helicity of the turbulence. It vanishes when: (i) the mean helicity of the turbulence is zero; (ii) the basic flow is uniform; (iii) the random process $\mathbf{V}^t$ is δ-correlated in time.

The derived equation is a nonlinear vector equation with coefficients that depend on the coordinates. The detailed investigation of this equation is a complicated problem. Here we restrict ourselves to an analysis of a linearised form of (3.10) with a specially chosen $\mathbf{V}^{(0)}$: a potential flow depending linearly on the coordinates. It turns out that exact solutions can be found in this case.

4. VORTEX GENERATION IN A POTENTIAL FLOW

Consider a basic flow $\mathbf{V}^{(0)}$ of the form.

$$\mathbf{V}^{(0)} = V_x^{(0)}(x)\mathbf{e}_x + V_y^{(0)}(y)\mathbf{e}_y + V_z^{(0)}(z)\mathbf{e}_z, \tag{4.1}$$

where $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are the unit coordinate vectors. This is a potential flow, $\nabla \times \mathbf{V}^{(0)} = 0$.

The incompressibility condition gives

$$\partial_x V_x^{(0)}(x) = -\partial_y V_y^{(0)}(y) - \partial_z V_z^{(0)}(z). \tag{4.2}$$

The right-hand side of (4.2) is independent of x; therefore $\partial V_x^{(0)}(x)/\partial x$ is uniform and $V_x^{(0)}(x)$ is a linear function of x. In a similar way we conclude that $V_y^{(0)}(y)$ and $V_z^{(0)}(z)$ are also linear functions of their arguments; expression (4.1) takes the form

$$\mathbf{V}^{(0)} = u_1 x\mathbf{e}_x + u_2 y\mathbf{e}_y + u_3 z\mathbf{e}_z, \tag{4.3}$$

where u_1, u_2 and u_3 are constants. Note that expression (4.3) can be considered as the Taylor expansion of an arbitrary velocity field near a saddle point. Due to incompressibility, $u_1+u_2+u_3=0$ and one or two of these constants are negative. The flow $\mathbf{V}^{(0)}$ given by (4.3) is an exact solution of the stationary Navier-Stokes equation

$$\partial_k(V_k^{(0)}V_i^{(0)}) = -\rho^{-1}\,\partial_i P^{(0)} + \nu_0 \Delta V_i^{(0)}, \tag{4.4}$$

for the pressure field

$$P^{(0)} = -\tfrac{1}{2}\rho|\mathbf{V}^{(0)}|^2 = -\tfrac{1}{2}\rho(u_1^2x^2 + u_2^2y^2 + u_3^2z^2). \tag{4.5}$$

The tensor G_{ijk} (3.11) takes a simple form for the linear flow $\mathbf{V}^{(0)}$:

$$G_{ijk} = 4G\,U_{im}\varepsilon_{mjk}, \tag{4.6}$$

where the matrix U is diagonal,

$$U_{im} \equiv \partial_i V_m^{(0)}(\mathbf{r}) = \begin{bmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{bmatrix}; \tag{4.7}$$

its trace vanishes due to the incompressibility condition. The averaged equation (3.10) also simplifies and when linearized in $\mathbf{V}^{(1)}$ becomes

$$\partial_t V_i^{(1)} + B\,\partial_k(V_i^{(0)}V_k^{(1)} + V_k^{(0)} + V_i^{(1)}) + 4G\,U_{im}\varepsilon_{mjk}\,\partial_j V_k^{(1)} = \nu\Delta V_i^{(1)} - \rho^{-1}\,\partial_i P^{(1)}. \tag{4.8}$$

Applying the curl-operator, we obtain the vorticity equation:

$$\partial_t \mathbf{\Omega} = B\nabla\times(\mathbf{V}^{(0)}\times\mathbf{\Omega}) - 4G\nabla\times(U\mathbf{\Omega}) + \nu\Delta\mathbf{\Omega}, \qquad \mathbf{\Omega} = \nabla\times\mathbf{V}^{(1)}. \tag{4.9}$$

The form of this equation coincides with the well-known dynamo-equation governing a large-scale magnetic field when an alpha-effect is present.

We follow Zeldovich *et al.* (1984) and seek a solution of (4.9) in the form of a plane wave in which both the amplitude and the wave vector depend on time:

$$\mathbf{\Omega}(t, \mathbf{r}) = \mathbf{w}(t, \mathbf{k}_0)\exp(i\mathbf{k}(t)\,.\,\mathbf{r}), \tag{4.10}$$

where $\mathbf{k}_0=\mathbf{k}(0)$ is the initial wavevector. Substitution of (4.10) into (4.9) gives, after equating terms of like power in $\mathbf{r}$,

$$\partial_t k_i = -BU_{ji}k_j, \tag{4.11}$$

$$\partial_t w_i = BU_{ij}w_j - 4iG\varepsilon_{ijm}k_jU_{mn}w_n - \nu k^2 w_i. \tag{4.12}$$

Let $u_1, u_2=\frac{1}{2}u>0$ and $u_3=-u<0$. It follows from (4.12) that the x- and y-components of the wave vector, k_x and k_y, decrease exponentially, while k_z increases in time as

$$k_z(t)=k_{0z}\,e^{But}, \tag{4.13}$$

This behavior is due to peculiar properties of the linear flow (4.3) under the incompressibility condition. In a linear flow any vector field should either be stretched along one axis and simultaneously compressed along the other axes to form a rope, or else be stretched along two axes and compressed along the remaining one to form a pancake (see Zeldovich *et al.*, 1984). The signature of the matrix U chosen here corresponds to the latter possibility. Thus, we set $k^2 \approx k_z^2$ in (4.12); the expression for the z-component of the vorticity follows as

$$w_z(t)=w_{0z}\exp[-But-\tfrac{1}{2}\nu Bu(k_z^2(t)-k_{0z}^2)], \tag{4.14}$$

where $w_{0z}=w_z(0)$. We see that $w_z(t)$ decreases first exponentially (when k_z is small) and then superexponentially (i.e. as an exponential function with an exponentially growing power). The x- and y-components of the solution are

$$w_x=w_y=w_0\exp[\tfrac{1}{2}But-\tfrac{1}{2}\nu Bu(k_z^2(t)-k_{0z}^2)]\cosh\left[\frac{2G}{B}\,(k_z(t)-k_{0z})\right], \tag{4.15}$$

for initial conditions $w_x(0)=w_y(0)=w_0$. It follows from this expression that the vorticity amplitude increases up until the moment

$$t_{gr}=\frac{1}{Bu}\ln\left\{\frac{Gu}{\nu k_{0z}}\left[1+\left(1+\frac{B\nu}{2G^2u}\right)^{1/2}\right]\right\}; \tag{4.16}$$

the smaller the u, the longer the period of growth.

We recall that (3.10) has been derived under the condition that the scale k^{-1} of the mean velocity fluctuation, $\mathbf{V}^{(1)}$, is much larger than the scale q^{-1} of the turbulence, $k\approx\gamma q$, where $\gamma\approx\lambda/L\ll 1$. But we see from (4.13) that k_z (and k) grows exponentially in time and finally this condition for self-consistency fails to be fulfilled. Equating k and αq, we obtain an estimate for the time up to which the (3.10) is applicable for the flow considered:

$$t<t_{ap}=\frac{1}{Bu}\ln\left(\frac{\gamma q}{k_{0z}}\right). \tag{4.17}$$

When the $But \ll 1$, the exponential function in $k_z(t)$ may be expanded in Taylor series:

$$k_z(t) \approx k_{0z}(1 + But + \ldots), \tag{4.18}$$

and the vorticity amplitude in the x, y-plane takes the form

$$w \approx w_0 \, e^{-\nu k_{0z}^2 + 2Guk_{0z})t} \tag{4.19}$$

Perturbations with $k = k_{\max}$, where $k_{\max} = Gu/\nu$ grow most rapidly. Their growth rate is $\Gamma_{\max} = (Gu)^2/\nu$. As we have agreed above, $k_{\max}$ should be small, $k_{\max} \approx \gamma q$, or $Gu/\nu \approx \gamma q$. Substitute this result into (4.16) and compare with (4.17) to obtain $t_{gr} \approx t_{ap}$, which means that, during all the period of applicability of equation (3.10), the vorticity does indeed grow.

The restriction $k_{\max} \ll q$ (which means that the structure has a large scale) imposes the restriction $Gu/(\nu q) \ll 1$ on the input parameters of the problem.

We have mentioned above the analogy between (4.9) and the dynamo equation for magnetic fields. Solutions (4.14) and (4.15) are also quite similar to solutions known in MHD. As shown by Gvaramadze *et al.* (1987, 1988), an analogous solution describes the evolution of magnetic field in a similar basic flow in MHD.

For the basic flow (4.3), equation (3.10) has been reduced to (4.9). We have derived an exact solution of (4.10) with the amplitude w described by (4.14) and (4.15). It follows from (4.15) that two trends occur simultaneously: the vorticity amplitude in the x, y-plane grows while the scale of the motion decreases in the z-direction. Equation (3.10), whose solution we now analyse, is valid only while the scales of the solution exceed the turbulent scale. Since one of them quickly decreases, our approximation soon becomes inapplicable. However, the amplitude of the solution grows faster than its scale decreases: the former increases exponentially with an exponent that itself grows exponentially, while the latter decreases exponentially. Hence, it becomes clear why the amplitude of the solution grows throughout the period of applicability of our theory. In other words, we can say that a large-scale vortex arises, one of whose scales decreases further and further, until it reaches the turbulent scale where our theory becomes inapplicable. This decrease in scale is associated with the form of (4.11), and its rate is determined by the form of the basic flow $\mathbf{V}^{(0)}$. Solution (4.19) is quite similar to the solution of the α^2-dynamo equation.

5. CONCLUSIONS

We have derived equation (3.10) that describes evolution of a large-scale velocity field in an incompressible fluid under the influence of a small-scale, homogeneous and isotropic helical turbulence. In this case, as for the compressible fluids considered by Moiseev *et al.* (1983a), the averaging procedure for the Navier-Stokes equation has led to the appearance of a new term that is proportional to the first derivatives of the velocity with respect to the coordinates. This term, which is similar to the alpha-effect term appearing in the dynamo equation of MHD, appears in the averaged equation because of the helicity of the turbulent velocity field, just as for the alpha-effect term in MHD.

Helicity is one of the most important characteristics of any vector field. First, the integral over the liquid volume $\int \mathbf{V} \cdot \nabla \times \mathbf{V}\, d^3r$ is an integral of motion for an ideal fluid i.e. it is an invariant. Second, this invariant characterizes the topology of a vector field and determines the number of linked streamlines. Third, a system with nonvanishing helicity lacks parity, i.e. it is not invariant under inversion of the coordinates, since the numbers of right-handed and left-handed vortices are not equal to each other.

The importance of helicity makes it interesting to carry out experimental research on large-scale vortex generation in helical turbulence. Unfortunately, we are not aware of any such experiments. For this reason, natural atmospheric turbulence proves to be a unique source of information about such turbulence. It is well known that the helicity of atmospheric turbulence is produced by the Coriolis force. The main candidate for the role of a large-scale structure associated with the helicity of atmospheric turbulence is the tropical cyclone (typhoon). This hypothesis was proposed by Moiseev *et al.* (1983b) on the basis of the following arguments. First, the characteristic spatial and temporal scales of a typhoon are much greater than the relevant scales of the atmospheric turbulence, which is believed to be causally connected with cyclogenesis. Second, the large-scale velocity field in a typhoon is helical. (The intense toroidal circulation in a tropical cyclone is linked to a weak poloidal one.) Third, the birth of tropical cyclones is often associated with the presence of a weak large-scale motion (e.g. an Eastern wave—see Riehl, 1976) which works as a trigger mechanism. Such motions may be interpreted as a regular basic flow which breaks the symmetry of the background state in the sense just described. As shown by Moiseev *et al.* (1983b), a mechanism based on the mean helicity of atmospheric turbulence furnishes an explanation of the energy transfer from small-scale convective motions to large-scale flows. Their estimates of the growth rate and characteristic scale of a growing atmospheric vortex are in qualitative agreement with the observed properties of typhoons. This approach offers an explanation of the different directions of air circulation in typhoons in the northern and southern hemispheres, gives a reasonable threshhold latitude for effective cyclogenesis, and even describes qualitatively the formation of the eye of a tropical cyclone.

Levich and Tzvetkov (1984, 1985) have proposed a mechanism of energy transfer from small-scale motions in the atmosphere to large scale motions typical of cyclones which is also based on the helical properties of turbulence. However, in contrast to our approach based on the non-vanishing mean helicity of the turbulence, the concept proposed by Levich and Tzvetkov depends on helicity fluctuations about states of vanishing mean helicity.

In our approach, because of the symmetry of the Reynolds stresses helicity alone is insufficient for the generation of large-scale vortices, but it turns out that the addition to the helical turbulence of even a steady potential flow is enough to engender large-scale vortices. The averaged equation is derived under the presumption that the turbulent Reynolds number is small, which is equivalent to a weak nonlinearity of the basic equations. However, in our opinion this restriction is not obligatory; it merely simplifies our calculations. As a matter of fact, the problem of the interaction of large-scale and small-scale processes inevitably involves small parameters, namely the ratios of the characteristic scales. It seems that the presence of these small parameters would permit the restriction to small Reynolds numbers to be lifted. We therefore believe that the closure of the basic equations can be performed for arbitrary Reynolds numbers, using the ratios of scales as perturbation parameters in the diagram technique; the arguments given by Tur *et al.* (1984) imply that the form of the averaged equation remains the same as in the case of small Reynolds numbers.

The averaged equation (3.10) has been analysed for a special regular basic flow, namely a potential flow, linear in the coordinates. Even this preliminary analysis has confirmed that the averaged equation has solutions that grow in time. These solutions describe large-scale vortex generation. The linear velocity field selected is very special, but can be understood as a local approximation to a smooth generic velocity field. Analysis of equation (3.10) for a more realistic regular basic flow will be thoroughly discussed elsewhere (see also Tur *et al.*, 1987).

Equation (3.10) describes the generation of large-scale vortices by helical turbulence, and has a form similar to the dynamo equation. This allows us to understand our results as a hydrodynamical analogue of the MHD alpha-effect or as a vortex dynamo.

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APPENDIX

The following term should be evaluated in (2.13):

$$\Pi_{ij}\,\partial_l[\langle V_l^1(\mathbf{r},t)\tilde{V}_j(\mathbf{r},t)\rangle+\langle \tilde{V}_l(\mathbf{r},t)\tilde{V}_j^1(\mathbf{r},t)\rangle], \tag{A.1}$$

Calculations are conveniently performed in **k**-space. Let us Fourier transform the spatial dependence:

$$\tilde{V}_i(\mathbf{k},t)=\frac{1}{(2\pi)^{3/2}}\int e^{-i\mathbf{k}\cdot\mathbf{r}}\,\tilde{V}_i(\mathbf{r},t)\,d^3r,$$

In **k**-space, (A.1) takes the form

$$i\Gamma_{lij}(\mathbf{k})[q_{lj}(\mathbf{k})+q_{jl}(\mathbf{k})], \tag{A.2}$$

where

$$\Gamma_{lij}(\mathbf{k})=\frac{1}{(2\pi)^{3/2}}k_l\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right),$$

$$q_{lj}(k)=\int\int d^3k_1d^3k_2\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\langle\tilde{V}_l(\mathbf{k}_1)V_j^t(\mathbf{k}_2)\rangle. \qquad \text{(A.3)}$$

The correlation $q_{ij}(\mathbf{k})$ can be easily calculated by using the Furutsu-Novikov formula (3.9). In **k**-space, the correlation function of the random field $\mathbf{V}^t$ has the form

$$\begin{aligned}\langle V_i^t(\mathbf{q},t)V_j^t(\mathbf{k}',t)\rangle &\equiv Q_{ij}(\mathbf{q},\tau)\delta(\mathbf{q}+\mathbf{k}')\\ &=[C(q,\tau)(\delta_{ij}-q_iq_j/q^2)+ig(q,\tau)\varepsilon_{ijl}q_e]\delta(\mathbf{q}+\mathbf{k}'),\end{aligned}$$

where $\tau\equiv t-t'$ is the correlation time of the turbulence. Insert this expression, together with the variational derivative (3.8) expressed in **k**-space, into (A.3) to obtain the following form for $q_{lj}(\mathbf{k})$:

$$\begin{aligned}q_{lj}(\mathbf{k},t)=&-\tfrac{1}{2}\int d\tau[V_k^{(1)}(\mathbf{k},t-\tau)\delta_{mn}+V_m^{(1)}(\mathbf{k},t-\tau)\delta_{kn}]\\ &\times\int d^3q\,e^{-\nu_0(\mathbf{k}-\mathbf{q})^2\tau}\,\Gamma_{klm}(\mathbf{k}-\mathbf{q})Q_{jn}(\mathbf{q},\tau)\\ &-\tfrac{1}{2}\int d\tau\int d^3k_1[V_k^{(0)}(\mathbf{k}_1)\delta_{mn}+V_m^{(0)}(\mathbf{k}_1)\delta_{kn}]\\ &\times[V_p^{(1)}(\mathbf{k}-\mathbf{k}_1,t-\tau)\delta_{ts}+V_t^{(1)}(\mathbf{k}-\mathbf{k}_1,t-\tau)\delta_{ps}]\\ &\times\int d^3q\frac{1}{\nu_0[(\mathbf{k}-\mathbf{q})^2-(\mathbf{k}-\mathbf{k}_1-\mathbf{q})^2]}[e^{-\nu_0(\mathbf{k}-\mathbf{k}_1-\mathbf{q})^2\tau}-e^{-\nu_0(\mathbf{k}-\mathbf{q})^2\tau}]\\ &\times\Gamma_{klm}(\mathbf{k}-\mathbf{q})\Gamma_{pnt}(\mathbf{k}-\mathbf{k}_1-\mathbf{q})Q_{js}(\mathbf{q},\tau),\end{aligned} \qquad \text{(A.4)}$$

where the integration over ξ has been performed.

The first term in (A.4) does not include the stationary flow velocity $\mathbf{V}^{(0)}$. After corresponding manipulations, this term produces the turbulent viscosity in (3.10). This turbulent viscosity is isotropic, i.e. it does not affect the scalar nature of the kinematic viscosity. Our expression for the isotropic turbulent viscosity coincides with the expression derived by Krause and Rüdiger (1974). The second term in (A.4) that contains $\mathbf{V}^{(0)}$ leads to appearance in (3.10) of completely new terms.

The following expressions appear in (A.4):

$$e^{-\nu_0(\mathbf{k}-\mathbf{q})^2\tau},$$

$$\Gamma_{pnt}(\mathbf{k}-\mathbf{q})=\frac{1}{(2\pi)^{3/2}}(k_p-q_p)\left[\delta_{nt}-\frac{(k_n-q_n)(k_t-q_t)}{|\mathbf{k}-\mathbf{q}|^2}\right],$$

Consider these terms more closely. The wave vector **k** refers to disturbances while the wave vector **q** refers to the turbulent pulsations. Since we are interested in large-scale disturbances of the velocity field we may set $|\mathbf{k}|\ll|\mathbf{q}|$: the scale of the disturbance greatly exceeds that of the turbulent pulsations. This allows us to expand the exponent and the denominator in $\Gamma_{pnt}(\mathbf{k}-\mathbf{q})$. In the following we consider only terms that include the first- and second-order derivatives of the large-scale disturbance. Therefore, we keep only the terms quadratic in $\mathbf{k}$ in the expansions of the exponent and denominator. These expansions have the form

$$e^{-\nu_0(\mathbf{k}-\mathbf{q})^2\tau} \approx e^{-\nu_0 q^2 t}[1-\nu_0 k^2\tau+2\nu_0(\mathbf{k}\cdot\mathbf{q})\tau+2\nu_0^2(\mathbf{k}\cdot\mathbf{q})^2\tau^2+\ldots]$$

$$\frac{1}{|\mathbf{k}-\mathbf{q}|^2}=\frac{1}{q^2}\left[1+\frac{2\mathbf{k}\cdot\mathbf{q}}{q^2}-\frac{k^2}{q^2}+\frac{4(\mathbf{k}\cdot\mathbf{q})^2}{q^4}+\ldots\right].$$

After these expansions are substituted into (A.4), the result can be integrated over the directions of the wave vector $\mathbf{q}$. Due to symmetry, the integrals which contain an odd number of $\mathbf{q}$ vectors vanish, while those which contain an even number of $\mathbf{q}$ factors are given by

$$\int q_i q_j \, d\Omega = (4\pi/3)q^2\,\delta_{ij},$$
$$\int q_i q_j q_k q_l \, d\Omega = (4\pi/15)q^4(\delta_{ij}\,\delta_{kl}+\delta_{ik}\,\delta_{jl}+\delta_{il}\,\delta_{jk}),$$

etc. Here $d\Omega$ is the solid angle element. After this angle integration, the construction of a symmetric combination $q_{lj}(k)+q_{lj}(k)$, the formation of the direct product with $i\Gamma_{lij}(k)$, and the transformation back to x-space, we obtain (3.10). Note that in (3.10) we omit terms which include anisotropic corrections to the turbulent viscosity associated with the stationary flow $\mathbf{V}^{(0)}$. These corrections are similar to the expression for the anisotropic viscosity derived by Shimomura and Yoshizawa (1986).